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Modified Quantum Mechanics and Generalized Uncertainty Principle

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Ci sarà sempre un nome, uno solo,

che appena lo senti, sorridi.

A Nicola

Contents

Abstract

The object of this thesis is the study of Modified Quantum Mechanics (i.e. the modifications of the Heisenberg canonical commutation relations) and the extension of Heisenberg's Uncertainty Principle, called Generalized Uncertainty Principle (GUP). In particular, I will focus on the application of the modified canonical commutation relations to the Harmonic Oscillator and discuss how HUP is modified in this context. Some frameworks (theoretical models) where GUP is derived are presented.

L'obiettivo del presente lavoro di tesi è lo studio della meccanica quantistica modificata e l'estensione del Principio D'Indeterminazione di Heisenberg, detto Principio D'Indeterminazione Generalizzato (GUP). In particolare, ci si concentrerà, sulle modifiche delle caniniche relazioni di commutazione dell'oscillatore armonico. Saranno discussi alcuni contesti dai quali il GUP emerge.

Preface

At the beginning of the 20th century some experiments showed that Classical Mechanics was not capable to describe reality so physicist needed a new theory to explain these "strange" empirical results.

This new theory, later called Quantum Mechanics, gradually arose from Max Planck's solution in 1900 to the black body radiation problem and Albert Einstein's 1905 paper explaining the photoelectric effect. Quantum Mechanics differs from Classical Physics in that quantities like energy, momentum and others are often restricted to discrete values (quantization), objects have characteristics of both particles and waves (wave-particle duality), and there are limits to the precision with which quantities can be known (uncertainty principle). In this way Classical Mechanics derives from Quantum Mechanics as an approximation valid only in the limit of macroscopic scales.

The modern formalization of Quantum Mechanics is based on some fundamental aspects:

- The state of a quantum system is described by a complex *wave function*, also referred to as unit state vector in a complex vector space (Hilbert Space);
- The wave function is solution of Schrödinger equation which tells how it evolves in time;
- The *statistical interpretation* which subsumes all of this and enables to figure out the possible results of any measurement, and their probabilities.

The Heisenberg Uncertainty Principle (HUP) represents one of the fundamental properties of quantum system. Accordingly to it, there should be a fundamental limit for the measurement accuracy, with which certain *pairs* of physical observables, such as the positions and momentum or energy and time, can not be simultaneously measured. In other words, the more precisely one observable is measured, the less precise the other one shall be detected.

The HUP don't give any limit to the precision on how an observable can be measured. This means that it's possible to assert that the particles is exactly in the position but have a completely undetermined momentum and vice versa. Recently, with the birth of the theories of *quantum gravity* or *string theory*, the idea that the HUP cannot be applied at all took place.

String Theory, *Quantum Geometry*, *Loop Quantum Gravity* and *Black Hole physic* all predict the existence of a *minimal length* of the order of Planck length scale. For example in the case of string theory it is conjectured that a particle described as a string does not interact at distances smaller than its size. As a consequence, the HUP has to be generalized to take into account this aspect.

The models which are designed to implement the minimal length scale and/or the maximum momentum in different physical systems entered the literature as the Generalized Uncertainty Principle (GUP).

In this thesis it will be studied a possible extension of Quantum Mechanics, the GUP and some applications. First of all I will recall some basic aspects of Quantum Mechanics, in particular the Heisenberg Uncertainty Principle.

Then I will illustrate the GUP and some applications, focusing on harmonic oscillator.

Chapter 1

Quantum Mechanics and Heisenberg's uncertainty principle

Consider a particle of mass m, constrained to move along the x-axis and subject to a specified potential $V(x,t)$. Quantum Mechanics approaches this problem looking for the particle's *wave function*, $\Psi(x,t)$, which is inferred by solving the *Schrödinger equation* [1]

$$\imath\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi \tag{1.1}$$

where $\imath$ is the imaginary unit and $\hbar$ is the Plank's constant divided by 2π. The Schrödinger equation plays a role logically analogous to Newton's second law: given suitable initial conditions (typically $\Psi(x,0)$) the Schrödinger equation determines $\Psi(x,t)$ for all future time, just as, in Classical Mechanics, Newton's law determines $x(t)$ for all future time given the conditions x_0 and p_0, the initial position and momentum of a particle respectively.

A particle, by its nature, is localized at a point, whereas the wave function (as its name suggests) is spread out in space (it's a function of x, for any given time t). How can such an object represent the state of a particle? The answer is provided by *Born's statistical interpretation* of the wave function, which says that $|\Psi|^2$ gives the probability of finding the particle at point x at the time t or, more precisely

$$\int_a^b |\Psi(x,t)|^2 dx = \begin{bmatrix} probability \quad of \quad finding \quad the \quad particle \\ between \quad a \quad and \quad b \quad at \quad time \quad t \end{bmatrix} \tag{1.2}$$

The statistical interpretation introduces a kind of *indeterminacy* into quantum mechanics.

It follows from eq.(1.2) that the integral in the whole space will be

$$\int |\Psi|^2 \, dx = 1 \tag{1.3}$$

Without this, the statistical interpretation would be nonsense. This condition is the so-called *normalization.* [1]

If the potential in eq.(1.1) is time independent it's possible to solve the Schrödinger equation by using the method of the *separation of variables*: we look for a solution that can be written as a simple product

$$\Psi(x,t) = \psi(x) f(t) \tag{1.4}$$

The separation of variables splits the partial differential equation into two ordinary differential equations

$$\frac{df}{dt} = -\frac{\imath E}{\hbar} f \tag{1.5}$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi \tag{1.6}$$

The first of these is easy to solve and the general solution is

$$f(t) = f_0 e^{-\imath Et/\hbar} \tag{1.7}$$

where f_0 is an arbitrary constant.

The equation (1.6) is called *time-independent Schrödinger equation.* This

[1] If the normalization condition does not occur, it is always possible to multiply the wave function for a constant in order to make it normalized

equation is used to determine the "spatial" part of wave function Ψ once the potential $V(x)$ is given.

1.1 Formalism

Quantum theory is based on two constructs: *wave functions* and *operators*. The state of a system is represented by its wave function, observables are represented by operators. Mathematically, the wave function is represented by abstract *vectors*, and operators act on them as *linear transformation*. So the natural language of quantum mechanics is *linear algebra*. [2]

Wave functions are represented by unit vectors in Hilbert space [3] [3].

Definition 1. Hilbert space $\mathbb{H}$ is a complex vector space together with an inner product with following properties

- $\langle v|u\rangle = \langle u|v\rangle^*$
- $\left\langle u|\alpha v + \alpha' v'\right\rangle = \alpha \langle u|v\rangle + \alpha' \left\langle u|v'\right\rangle$
- $\langle v|v\rangle \geq 0 \quad \wedge \quad \langle v|v\rangle = 0 \Leftrightarrow v = 0$

$\forall u, v, v' \in \mathbb{H} \quad \wedge \quad \forall \alpha, \alpha' \in \mathbb{C}$

Hilbert space is also a complete metric space with respect to the inner product induced metric.

Linear Transformation, T, are represented by *matrices*, which act on vectors (to produce new vectors) by the ordinary rules of matrix multiplication:

[2] Linear algebra is the branch of mathematics concerning vector spaces and linear mappings between such spaces. It includes the study of lines, planes, and subspaces, but is also concerned with properties common to all vector spaces.

[3] In quantum mechanics, *wave functions live in Hilbert spaces*. The set of all *square-integrable functions*, on a specified interval

$$f(x) \qquad \text{such that} \qquad \int_a^b |f(x)|^2 dx < \infty \tag{1.8}$$

constitutes a vector space. Physicist call it L^2(a,b) (square integrable functions)

$$|\beta\rangle = T\,|\alpha\rangle \rightarrow \mathbf{b} = \mathbf{Ta} = \begin{pmatrix} t_{11} & \cdots & t_{1N} \\ \vdots & \ddots & \vdots \\ t_{N1} & \cdots & t_{NN} \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_N \end{pmatrix} \tag{1.9}$$

But the "vectors" we encounter in quantum mechanics are $functions$, and they live in *infinite-dimensional* spaces. For them the N-tuple/matrix notation is awkward, and manipulations that are well-behaved in the finite-dimensional case can be problematic.

We can define some properties about the operators:

- *The inverse of an operator.* Given a linear operator T, we can define its inverse, denoted by T^{-1}, if, and only if, the correspondence defined by T is one-to-one, i. e. if $Tv \neq Tu \Leftrightarrow v \neq u$. If the condition is satisfied, we define T^{-1} by:

$$T^{-1}Tv = TT^{-1}v = v \quad \forall v \qquad i.e. \quad T^{-1}T = TT^{-1} = I \tag{1.10}$$

- *Sum of operators.* The sum of two operators T, S is defined as

$$(T + S)\,v = Tv + Sv \quad \text{with} \quad v \quad \text{endomorfism of} \quad \mathbb{H} \tag{1.11}$$

- *Product of two operators.* The product of two operators is defined as

$$TSv = T\,(Sv) \quad \text{with} \quad v \quad \text{endomorfism of} \quad \mathbb{H} \tag{1.12}$$

Note that TS is, in general, different by ST (the operators T and S in general do not commute).

We can define, for any operator, the adjoint operator. An operator that coincides with its adjoint is called a selfadjoint operator. In the finite-dimensional case, the concept of selfadjoint operator coincides with that of Hermitian operator.

- *Adjoint operator.* We can associate to each operator T on $\mathbb{H}$ its adjoint operator (which we will denote in the following with $T^\dagger$) through the condition that,

$$\langle Tv|u\rangle = \left\langle v|T^\dagger u\right\rangle \quad \text{with} \quad v,u \quad \text{endomorfisms of} \quad \mathbb{H} \tag{1.13}$$

- *Selfadjoint operator.* An operator A on $\mathbb{H}$ is an Hermitian (or symmetric) operator if, $\forall \quad v,u \quad$ endomorfisms of $\quad \mathbb{H}$, it satisfies

$$\langle Av|u\rangle = \langle v|Au\rangle \tag{1.14}$$

  This means that $A = A^\dagger$, and the operator is also said selfadjoint.

In Classical Mechanics, the total energy (kinetics plus potential) is called the *Hamiltonian*.

$$H(x,p) = \frac{p^2}{2m} + V(x) \tag{1.15}$$

The corresponding Hamiltonian *operator*, obtained by the canonical substitution $p \to \frac{\hbar}{\imath}\frac{\partial}{\partial x}$ is therefore

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x) \tag{1.16}$$

Thus, the time-independent Schrödinger equation (1.15), can be written as

$$\hat{H}\Psi = E\Psi \tag{1.17}$$

Ordinarily, when an observables Q is measured on an ensemble of identically prepared systems, all in the same state Ψ, we can not get the same results each time - this is the *indeterminacy* of quantum mechanics. But it is possible to prepare a state such that every measurement of Q is certain to return the same value (call it q). This is a *determinate state* for the observable Q.

Well, the standard deviation of Q, in a determinate state, would be *zero*

$$\sigma^2 = \left\langle \left(\hat{Q} - \langle Q \rangle\right)^2 \right\rangle = \left\langle \Psi | \left(\hat{Q} - q\right)^2 \Psi \right\rangle = \left\langle \left(\hat{Q} - q\right) \Psi | \left(\hat{Q} - q\right) \Psi \right\rangle = 0 \tag{1.18}$$

If every measurement gives q, their average is also q: $\langle Q \rangle = q$. We then obtain

$$\hat{Q}\Psi = q\Psi \tag{1.19}$$

This is an *eigenvalue equation* for the operator $\hat{Q}$; Ψ is an *eigenfunction* of $\hat{Q}$, and q is the corresponding *eigenvalue*. For example, some states of the total energy are eigenfunction of the Hamiltonian (1.17)

$$\hat{H}\Psi = E\Psi \tag{1.20}$$

where E is the eigenvalue, and Ψ is the eigenfunction.

1.2 Heisenberg uncertainty principle

In Quantum Mechanics, an observable is a dynamic variable that can be measured. In the mathematical approach of quantum mechanics, an observable is represented by hermitian linear operator that acts on a vector of the system. In general, linearity is expressed by:

$$\hat{\mathbf{f}} \left(\mathbf{c_1}\psi_{\mathbf{1}} + \mathbf{c_2}\psi_{\mathbf{2}}\right) = \mathbf{c_1}\hat{\mathbf{f}}\psi_{\mathbf{1}} + \mathbf{c_2}\hat{\mathbf{f}}\psi_{\mathbf{2}} \tag{1.21}$$

Quantum Mechanics is intrinsically probabilistic, and this is quantitatively described by the Heisenberg Uncertainty Principle. Quantum Mechanic allows to predict the behaviour of a quantum system based on the probability of finding a certain value of the observable. A measure causes the wave function, in general written as an infinite superposition of states, to collapse on an eigenstate of the observable. This leads to the fact that all possible values that can assume an observable must be eigenvalues of the observable itself.

A state of a system can be written as

$$|\varphi\rangle = \sum_j c_j |e_j\rangle \tag{1.22}$$

where $|e_j\rangle$ are basis vectors. The action of the observer A on this state is full identify by his action on this vectors.

$$\mathbf{A}|\varphi\rangle = \sum_j c_j \mathbf{A}|e_j\rangle = \sum_j c_j \sum_i c_{ij}^A |e_i\rangle = \sum_{i,j} c_j c_{ij}^A |e_i\rangle \tag{1.23}$$

where c_{ij}^A are the eigenvalues of the operator A.

$$c_{ij}^A = \langle e_i|A|e_j\rangle \tag{1.24}$$

The *average value of an observable* is defined as

$$\langle A\rangle = \langle\varphi|A|\varphi\rangle = \int_{-\infty}^{+\infty} \varphi^* A \varphi \; dx \tag{1.25}$$

Given an observables A, we define the standard deviation of the operator A as

$$\triangle A = A - \langle A\rangle \tag{1.26}$$

For any observable A, we have

$$\sigma_A^2 = \langle (A - \langle A\rangle)\,\Psi \,|\, (A - \langle A\rangle)\,\Psi\rangle = \langle f|f\rangle \tag{1.27}$$

where $f \equiv (A - \langle A\rangle)\,\Psi$. σ^2 is called *variance*. Similarly for the observable, B,

$$\sigma_B^2 = \langle g|g\rangle\,, \qquad \text{where} \qquad g \equiv (B - \langle B\rangle)\,\Psi \tag{1.28}$$

Therefore, from the Schwartz inequality, on gets

$$\sigma_A^2 \sigma_B^2 = \langle f|f\rangle \langle g|g\rangle \geq |\,\langle f|g\rangle\,|^2 \tag{1.29}$$

Now, for any complex number z one gets

$$|z|^2 = |Re\,(z)\,|^2 + |Im\,(z)\,|^2 \geq [Im\,(z)]^2 = \left[\frac{1}{2\imath}\,(z - z^*)\right]^2 \tag{1.30}$$

Setting $z = \langle f|g\rangle$

$$\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2\imath}\,[\langle f|g\rangle - \langle g|f\rangle]\right)^2 \tag{1.31}$$

Explicit calculations of $\langle f|g\rangle$ give

$$\begin{aligned}\langle f|g\rangle &= \langle (A - \langle A\rangle)\,\Psi|\,(B - \langle B\rangle)\,\Psi\rangle = \langle \Psi|\,(A - \langle A\rangle)\,(B - \langle B\rangle)\,\Psi\rangle = \\ &= \left\langle \Psi|\,\left(A\hat{B} - A\,\langle B\rangle - B\,\langle A\rangle + \langle A\rangle\,\langle B\rangle\right)\,\Psi\right\rangle = \\ &= \langle \Psi|AB\Psi\rangle - \langle B\rangle\,\langle \Psi|A\Psi\rangle - \langle A\rangle\,\langle \Psi|B\Psi\rangle + \langle A\rangle\,\langle B\rangle\,\langle \Psi|\Psi\rangle = \\ &= \langle AB\rangle - \langle B\rangle\,\langle A\rangle - \langle A\rangle\,\langle B\rangle + \langle A\rangle\,\langle B\rangle = \\ &= \langle AB\rangle - \langle A\rangle\,\langle B\rangle\end{aligned} \tag{1.32}$$

In the same way,

$$\langle g|f\rangle = \langle BA\rangle - \langle A\rangle\,\langle B\rangle \tag{1.33}$$

So

$$\langle f|g\rangle - \langle g|f\rangle = \langle AB\rangle - \langle BA\rangle = \langle [A, B]\rangle \tag{1.34}$$

where

$$[A, B] = AB - BA \tag{1.35}$$

is the commutator of two operators. Equation (1.31) assumes the form

$$\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2\imath}\,\langle [A, B]\rangle\right)^2 \tag{1.36}$$

This is the *uncertainty principle*. The imaginary unit $\imath$ is not a problem because if one of the observable is the position and the another one is momentum, their commutator is

$$[x, p] = \imath\hbar \tag{1.37}$$

and the imaginary unit will disappear. The *uncertainty principle* can be written as :

$$\Delta x^2 \Delta p^2 \geq \left(\frac{1}{2\imath}\imath\hbar\right)^2 = \left(\frac{\hbar}{2}\right)^2 \tag{1.38}$$

$$\Delta x \Delta p \geq \frac{\hbar}{2} \tag{1.39}$$

1.3 Harmonic Oscillator, annihilation and creation operators

In quantum mechanics, the quantum harmonic oscillator is the treatment of a system characterized by an harmonic potential. This is one of the most important problems in theoretical physics, since every potential can be approximated to an harmonic potential around a point of stable equilibrium. Now we are going to write Schrödinger equation (1.17) with the potential

$$V = \frac{1}{2} m\omega^2 x^2 \tag{1.40}$$

The Schrödinger equation reads

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2} m\omega^2 x^2 \psi = E\psi \tag{1.41}$$

We can find two different approaches to this problem. The first is *series method* and the second is the so-called *algebraic method*. We shall analyze the latter.

1.3.1 Algebraic Method

Let's write the Hamiltonian (1.41) in a more suggestive form

$$\frac{1}{2m}\left[p^2+(m\omega x)^2\right]\psi = E\psi \Leftrightarrow H\Psi = E\Psi \tag{1.42}$$

where $p \equiv \left(\frac{\hbar}{\imath}\right)\frac{d}{dx}$ is the momentum operator. I recall that for simplicity I omit the superscribed hats indicating operators.

The Hamiltonian is

$$H = \frac{1}{2m}\left[p^2+(m\omega x)^2\right] \tag{1.43}$$

Here p and x are *operators*, that do not *commute* eq. (1.37).

Let's introduce the quantities

$$a_\pm \equiv \frac{1}{\sqrt{2\hbar m\omega}}\left(\mp\imath p + m\omega x\right) \tag{1.44}$$

Notice that usual $a_+ = a^\dagger$ and $a_- = a$.

The product a_-a_+ is

$$\begin{aligned} a_-a_+ = \frac{1}{2\hbar m\omega}\left(\imath p + m\omega x\right)\left(-\imath p + m\omega x\right) = \\ \frac{1}{2\hbar m\omega}\left[p^2+(m\omega x)^2 - \imath m\omega\left(xp - px\right)\right] \end{aligned} \tag{1.45}$$

The extra term, $(xp-px)$ is the *commutator* of x and p. We therefore get

$$a_-a_+ = \frac{1}{2\hbar m\omega}\left[p^2+(m\omega x)^2\right] - \frac{\imath}{2\hbar}\left[x,p\right] = \frac{1}{\hbar\omega}H + \frac{1}{2} \tag{1.46}$$

The latter equation implies

$$H = \hbar\omega\left(a_-a_+ - \frac{1}{2}\right) \tag{1.47}$$

In the same way

$$a_+ a_- = \frac{1}{\hbar\omega} H - \frac{1}{2} \tag{1.48}$$

from which we get

$$H = \hbar\omega \left(a_+ a_- + \frac{1}{2} \right) \tag{1.49}$$

The commutator (1.37) implies that the operators $a_\pm$ defined in (1.44) satisfy the commutation relations

$$[a_-, a_+] = 1 \tag{1.50}$$

In therms of $a_\pm$ the Schroedinger equation for the harmonic oscillator takes the form

$$\hbar\omega \left(a_\pm a_\mp \pm \frac{1}{2} \right) \psi = E\psi \tag{1.51}$$

We call $a_\pm$ *ladder operators*, because they allow to climb up and down in energy; a_+ is the *raising operator* and a_- the *lowering operator*. Apply the lowering operator repeatedly, we obtain a state with energy less than zero, which does not exist. Then we impose

$$a_- \psi_0 = 0 \tag{1.52}$$

where ψ_0 is lowest rung state (fundamental state). We can use the latter equation to determine $\psi_0(x)$

$$\frac{1}{\sqrt{2\hbar m\omega}} \left(\hbar \frac{d}{dx} + m\omega x \right) \psi_0 = 0 \tag{1.53}$$

Solving this equation by using separation of variables we get

$$\psi_0(x) = A\, e^{-\frac{m\omega}{2\hbar} x^2} \tag{1.54}$$

Finally, normalizing the wave function, we obtain

$$\psi_0(x) = \sqrt[4]{\frac{m\omega}{\pi\hbar}}\; e^{-\frac{m\omega}{2\hbar}x^2} \tag{1.55}$$

The energy corresponding to the fundamental level is

$$E_0 = \frac{1}{2}\hbar\omega \tag{1.56}$$

Applying repeatedly the raising operator we obtain

$$\psi_n(x) = A_n (a_+)^n \psi_0(x) \;\text{ with }\; E_n = \left(n + \frac{1}{2}\right)\hbar\omega \tag{1.57}$$

where A_n is the normalization constant.

Chapter 2

Generalized Uncertainty Principle and Quantum Mechanics

String theory, double special relativity, and quantum gravity are examples of modern approaches in which the existence of a minimal length of the order of Planck length could arise. This feature leads to a modification of *Heisenberg Uncertainty Principle*. Such a modified Heisenberg uncertainty principle is referred in literature as a *Generalized Uncertainty Principle* (GUP). Here, we study some aspects of GUP in the spirit of Quantum Mechanics and the frameworks where GUP arises.

2.1 String Theory

In String Theory, a GUP result was the first derive in [16]. The ultra high-energy scatterings of Strings were studied in order to check how the theory tackles in consistency of Quantum Gravity at the Planck scale. Some interesting effects are compared to those which were found in usual field theories, especially the ones originating from the soft short- distance behaviour of the String theory [13]. Processes are studied at a short distance as the high-energy fixed-angle scatterings. The latter are apparently not able to test distances shorter than the characteristic String length $\lambda_s = (\hbar\alpha)^{\frac{1}{2}}$, where α is the String tension. Another scale is dynamically generated. The D- dimensional gravitational Schwarzschild radius $R(E) \sim (G_N E)^{\frac{1}{D-3}}$ approaches toward the String length λ_s. This depends on whether $R(E)$ is smaller or greater than λ_s. If $R(E) > \lambda_s$, appear new contributions at distances of the order of $R(E)$. This indicates a classical gravitational instability that can be attributed to the black hole formation. Instead, if $R(E) < \lambda_s$ the contributions are irrelevant. There are no black holes with a radius smaller than the String length, and the analysis of short distance can go on. The analysis of the angle distance relationship suggests the existence of a scattering angle θ_M. When the scattering should take place at $\theta < \theta_M$, the relation between the interaction distance and the momentum transfer is the classical one. But, when $\theta \gg \theta_M$ the classical picture is no longer valid. This suggests a modification of the uncertainty relation at the Planck scale

$$\triangle x \sim \frac{\hbar}{\triangle p} + Y\alpha\triangle p \tag{2.1}$$

where Y is a suitable constant.

2.2 Doubly special relativity

The doubly relativistic theory [13] is a group of transformations with two invariants. In additional to the constant speed of light, it is also assumed that an invariant energy scale should exist. Nevertheless, this group of transformation remains Lorentzian. A nonlinear realization of the Lorentz transformations in energy- momentum (E, p) space parametrized by an invariant length l

$$\epsilon = Ef\left(lE, l^2p^2\right) \tag{2.2}$$

$$\pi_i = P_i g\left(lE, l^2p^2\right) \tag{2.3}$$

where (ϵ, π) are auxiliary transforming variables which define the nonlinear Lorentz transformation of the physical energy-momentum (E, p). Then, we get two functions with two variables f and g. These functions are able to parametrize more general nonlinear realization of the Lorentz transformations with rotations realized as linearly depending on the dimensional scale. The condition to recover the special relativistic theory at low energy reduces to the condition $f\,(0;0) = g\,(0;0) = 1$. The choice of the two function f and g will lead to a generalization of the relativity principle with an invariant length scale. The Lorentz transformations connecting the energy-momentum of a particle in different inertial frames differ from the standard special relativistic linear transformations which are recovered when $lE \ll 1$ and $l^2p^2 \ll 1$. A nonlinear realization of the Lorentz transformations corresponds to the choice of the two functions

$$f = \frac{1}{2}\left[\left(1 + l^2p^2\right)\frac{e^{lE}}{lE} - \frac{e^{-lE}}{lE}\right] \tag{2.4}$$

$$g = e^{lE} \tag{2.5}$$

For a particle of mass m, the relation between the energy and momentum is given by

$$\left(1 - l^2p^2\right)e^{lE} + e^{-lE} = e^{lm} + e^{-lm} \tag{2.6}$$

where the quantity e^{lE} is given by

$$e^{lE} = \frac{\cosh\left(lm\right) + \sqrt{\cosh^2\left(lm\right) - \left(1 - l^2p^2\right)}}{\left(1 - l^2p^2\right)} \tag{2.7}$$

An upper bound on the momentum can be defined as

$$p^2_{max} < \frac{1}{l^2} \tag{2.8}$$

suggesting the existence of a minimal measurable length which would restrict the momentum of the test particle to take any arbitrary value. This leads to an upper bound, P_{max}. The commutation relation between the canonical variables x and p read

$$[X_i, P_j] = \imath\hbar \left[e^{-lE}\delta_{ij} + \frac{l^2}{\cosh{(lm)}} p_i p_j \right] \tag{2.9}$$

This result suggests the possibility to relate the transition from the quantum behaviour at the microscopic level to the classical behaviour at the macroscopic level with the modification of quantum mechanics induced by a modification of the relativity principle.If we consider the massless particle, the value of e^{lE} is

$$e^{lE} = \frac{1}{1 - l|p|} \tag{2.10}$$

It is found that the commutation relation between the canonical variables x and p should be modified in doubly special relativity as

$$[X_i, P_j] = \imath\hbar \left[(1 - l|p|)\, \delta_{ij} + l^2 p_i p_j \right] \tag{2.11}$$

It is apparent that when the momentum approaches its maximum value, one has a nontrivial limit for the canonical commutation relation. Definition of the algebra leads to a GUP, as we will discuss in the next sections.

2.3 Black hole physics

Several works have been devoted to perform the uncertainty relations and their measurability bounds in Quantum Gravity [13]. Gedanken experiments have been proposed to measure the area of apparent horizon of a black hole. According to modern theories ,for example quantum gravity HUP breaks down for energies close to the Planck scale where the corresponding Schwarzschild radius [1] becomes comparable with the Compton wavelength and both becoming approximately equal to the Plack length. In String theories, the tool of gedanken String collisions at Planck energy was very useful. The black hole

[1] The Schwarzschild radius is the radius of a sphere such that, if all the mass of an object were to be compressed within that sphere, the escape velocity from the surface of the sphere would equal the speed of light. If a stellar remnant were to collapse to or below this radius, light could not escape and the object is no longer directly visible outside, thereby forming a black hole.

approach to GUP is agree, especially in its functional form, with what is just obtained in the framework of the String Theory. The gedaken experiment proceeds by observing the photons scattered by the studied black hole. The main physical hypothesis of the experiment is that the black hole emits Hawking radiation. Detecting the Hawking radiation, it turns to be possible to span an "image" of the black hole. Besides, measuring the direction of the propagating photons that are emitted at different angles and tracing them back, we can locate the position of the center of the hole. As a consequence of this analysis a GUP is recovered.

2.4 GUP with a minimal length

In ordinary quantum mechanics, the standard Heisenberg Uncertainty Principle is given by

$$\triangle x \triangle p \geq \frac{\hbar}{2} \tag{2.12}$$

Generally, the uncertainty principle can be formulated as

$$\triangle x \triangle p \geq \frac{\hbar}{2} + \beta_0 l_{pl}^2 \frac{(\triangle p)^2}{\hbar} \tag{2.13}$$

The additional term,$\beta_0 \left(l_{pl}^2 \frac{(\triangle p)^2}{\hbar} \right)$ has its origin on the nature of spacetime at the Planck energy scale. The simplest GUP relation which implies the appearance of a nonzero minimal uncertainty $\triangle x_0$ in position has the form

$$\triangle x \triangle p \geq \frac{\hbar}{2} \left(1 + \beta \left(\triangle p \right)^2 + \beta \left\langle p \right\rangle^2 \right) \tag{2.14}$$

where β is the GUP parameter defines as $\beta = \frac{\beta_0}{M_{pl} c^2} = \beta_0 \frac{l_{pl}^2}{\hbar^2}$ and $M_{pl} c^2 \approx 10^{19} GeV$ is the 4-dimensional fundamental Planck scale. It is normally assumed that β_0 is not far from unity.

While in ordinary quantum mechanics $\triangle x$ can be made arbitrarily small by letting $\triangle p$ grow correspondingly, this is no longer the case if eq. (2.14) holds. If for decreasing $\triangle x$, $\triangle p$ increases, the new term $\beta \left(\triangle p \right)^2$ on the right

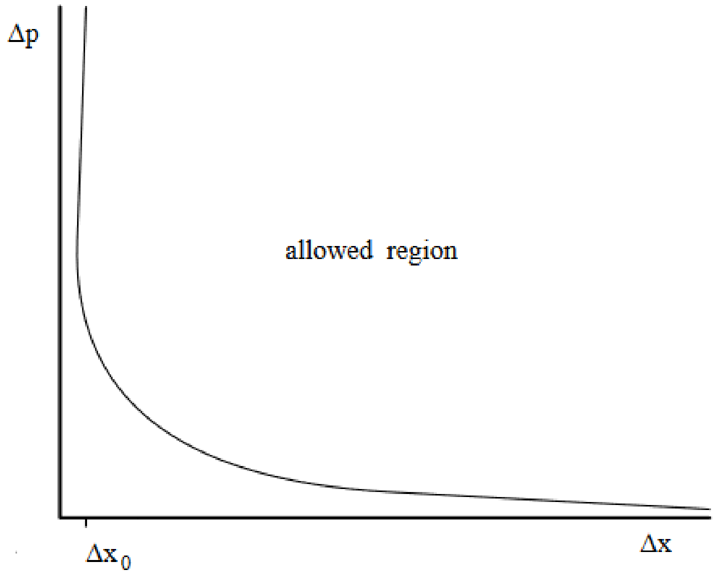

Figure 2.1: Modified uncertainty relation, implying a 'minimal length' $\triangle x_0 > 0$

of eq. (2.14) will eventually grow faster then the left side. Hence $\triangle x$ can no longer be made arbitrarily small.

At energies much below the Planck energy, the extra term in (2.14) would be irrelevant,which means $\beta \to 0$ and the standard HUP relation is recovered. For any pair of observables **A** and **B**

$$\triangle A \triangle B \geq \frac{\hbar}{2} \ |\langle [A, B] \rangle| \tag{2.15}$$

Typically to derive the GUP (2.13) or (2.14) one modifies the algebraic structure canonical commutation relation as

$$[x, p] = \imath \hbar \left(1 + \beta p^2\right) \tag{2.16}$$

and we define position and momentum operators for the GUP case as

$$X = x \tag{2.17}$$

$$P = p \left(1 + \beta p^2\right) \tag{2.18}$$

here x and p ensure the Jacobi identities, $[x_i, p_j] = \imath\hbar\delta_{ij},\ [x_i, x_j] = [p_i, p_j] = 0$. We interpret p as the momentum operator at low energies which as the representation $p_j = \frac{\hbar}{\imath}\frac{\partial}{\partial x_j}$, and P as the momentum operator at high energies, where has the generalized representation $P_j = \frac{\hbar}{\imath}\frac{\partial}{\partial x_j}\left[1 + \beta\left(\frac{\hbar}{\imath}\frac{\partial}{\partial x_j}\right)^2\right]$.

2.5 GUP with minimal length and Maximal momentum

In general GUP with minimal length and maximal momentum predicts that a test particles momentum cannot be arbitrarily imprecise and therefore there is an upper bound for momentum uncertainty. This lead to a maximal measurable momentum for a test particle. The GUP that predicts both a minimal observable length and a maximal momentum can be written in the form

$$\triangle x \triangle p \geq \frac{\hbar}{2}\left(1 - 2\alpha\left\langle p\right\rangle + 4\alpha^2\left\langle p^2\right\rangle\right) \tag{2.19}$$

where $\alpha = \frac{\alpha_0}{M_{pl}c} = \alpha_0\frac{l_{pl}}{\hbar}$. Lets define

$$X = x \tag{2.20}$$

$$P = p\left(1 - \alpha p + 2\alpha^2 p^2\right) \tag{2.21}$$

where x and p satisfy the canonical commutation relations via the Jacobi identity, and X and P satisfy the generalized commutation relation

$$[X.P] = \imath\hbar\left(1 - \alpha p + 2\alpha^2 p^2\right) \tag{2.22}$$

Here there is an extra,first order term in particle's momentum which has its origin on the existence of a maximal momentum. Let's show how maximal momentum arises in this setup. The absolute minimal measurable length is given by $\triangle x_{min}(\langle p\rangle = 0) \equiv \triangle x_0 = 2\alpha\hbar$ see equation (A.4).

Consider eq. (2.19) in the boundary of the allowed region. Setting $\langle p\rangle = 0$ (to obtain the absolute maximal momentum) we find

$$\triangle x \triangle p = \frac{\hbar}{2}\left(1 + 4\alpha^2 \left\langle p^2 \right\rangle\right) \tag{2.23}$$

Since $\triangle p = \sqrt{\left\langle p^2 \right\rangle - \left\langle p \right\rangle^2} = \sqrt{\left\langle p^2 \right\rangle}$, we find

$$\triangle x \triangle p = \frac{\hbar}{2}\left(1 + 4\alpha^2 \left(\triangle p\right)^2\right) \tag{2.24}$$

This results in

$$\left(\triangle p\right)^2 - \frac{\triangle x}{2\alpha^2 \hbar}\triangle p + \frac{1}{4\alpha^2} = 0 \tag{2.25}$$

So replacing $\triangle x \to \triangle x_{min}$ and $\triangle p \to \triangle p_{max}$ we get

$$\left(\triangle p_{max}\right)^2 - \frac{\triangle x_{min}}{2\alpha^2 \hbar}\triangle p_{max} + \frac{1}{4\alpha^2} = 0 \tag{2.26}$$

Using the value of $\triangle x_{min} = 2\alpha\hbar$, we find

$$\left(\triangle p_{max}\right)^2 - \frac{1}{\alpha}\triangle p_{max} + \frac{1}{4\alpha^2} = 0 \tag{2.27}$$

The solution of this equation is

$$\triangle p_{max} = \frac{1}{2\alpha} \tag{2.28}$$

which is the maximal momentum.

2.6 Kampf model and noncommutativity model

In this last section we shall discuss another framework in which the GUP and the modified canonical commutation relations arises. This is the non commutative algebra. In some sense it is a generalization of what we already discussed in previous section. Due to the complexity of the argument I will just give a short review of the idea. Has we have seen, HUP actually has a strong relationship to the canonical commutation or commutative phase space structures. When HUP should be broken down by GUP, an operational form of noncommutative (NC) phase space structures will be observed. i.e.

$$[x_i, p_j] = \imath\hbar \left[\delta_{i,j}\left(1 + \beta f_1\left(p^2\right)\right) + f_2\left(p^2\right) p_i p_j\right] \tag{2.29}$$

$$[x_i, x_j] = \imath\hbar f_{ij}\left(p\right) \neq 0 \tag{2.30}$$

The presence of a minimum length scale or a maximum momentum scale or both simply leads to the possibility of originating GUP with NC algebra. Both are likely consistent. Based on this, Kampf proposed the following algebraic relations [9]

$$[x_i, p_j] = \imath\hbar \left[\delta_{ij}\left(1 + \beta p^2\right) + \beta^{'} p_i p_j\right] \tag{2.31}$$

$$[x_i, x_j] = \imath\hbar \left(\beta^{'} - 2\beta\right)\left(x_i p_j - x_j p_i\right) \tag{2.32}$$

$$[p_i, p_j] = 0 \tag{2.33}$$

Other algebraic relations are [9]

$$[x_i, p_j] = \imath\hbar\delta_{ij}\left(1 + \beta p^2\right) \tag{2.34}$$

$$[x_i, x_j] = -2\imath\hbar\beta\left(x_i p_j - x_j p_i\right) \tag{2.35}$$

$$[p_i, p_j] = 0 \tag{2.36}$$

Finally we report a new algebraic relations recently proposed [9]

$$[x_i, p_j] = \imath\hbar \left[\delta_{ij}\left(1 + \beta p^2\right) + \beta^{'} p_i p_j + O\left(\beta^{'2}, \beta^2\right)\right] \tag{2.37}$$

$$[x_i, x_j] = \imath\hbar \frac{\left(2\beta - \beta^{'}\right) + \left(2\beta + \beta^{'}\right)\beta p^2}{1 + \beta p^2}\left(x_i p_j - x_j p_i\right) \tag{2.38}$$

$$[p_i, p_j] = 0 \tag{2.39}$$

It is evident that the GUP approaches, which are consistent with NC algebras, can open the possibility of space discreteness and quantization. i.e. , the physical states of space should be noncommutative. Noncommutativity of spacetime is characterized by the general expression for the commutation of the variable x^μ [14]

$$[x^\mu, x^\nu] = \imath\theta^{\mu\nu} \tag{2.40}$$

where $\theta^{\mu\nu}$ is an anti-symmetric matrix determining the fundamental discretization and quantization of the phase space. For detail see the excellent review [14]

Chapter 3

GUP Applications

In this chapter we will recall the Harmonic Oscillator set in Ch.1 and we will see the consequences of the changes due to the GUP.

3.1 Harmonic Oscillator in GUP

Now we are going to show dynamics and quantum mechanical coherent states of a simple harmonic oscillator. Equations of motion for simple harmonic oscillator are derived and some of their new implications are discussed. Then coherent states of harmonic oscillator in the case of GUP are compared with relative situation in ordinary Quantum Mechanics. It is shown that in the framework of GUP there is no considerable difference in definition of coherent states relative to ordinary Quantum Mechanics, but considering expectation values and variances of some operators based on quantum gravitational arguments one concludes that although it is possible to have complete coherency and vanishing broadening in usual Quantum Mechanics, gravitational induced uncertainty destroys complete coherency in quantum gravity.

3.1.1 Dynamics harmonic oscillator

In Heisenberg picture of Quantum Mechanics, equation of motion for an observable A is

$$\frac{dA}{dt} = \frac{\imath}{\hbar}[H, A] \tag{3.1}$$

Hamiltonian for a simple harmonic oscillator is

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 \tag{3.2}$$

Using eq.(3.1) and commutation relations (2.35) and (2.34) we find that the equation of motion for x and p are respectively

$$\frac{dx}{dt} = \frac{1}{m}\left(p + \beta p^3\right) \tag{3.3}$$

and

$$\frac{dp}{dt} = \frac{1}{2}m\omega^2\left(2x + \beta x p^2 + \beta p^2 x\right) \tag{3.4}$$

Using Baker - Hausdorff lemma (see appendix B), a lengthy calculation gives the following equations for time evolution of x and p [15]

$$\begin{aligned}
x(t) = x(0)\cos\omega t + \frac{p(0)}{m\omega}\sin\omega t \qquad (3.5)\\
+\beta\left[\frac{p^3(0)}{m\omega}(\omega t) - \frac{1}{2}\left(p(0)x(0)p(0) + \frac{3}{2}\left[x(0)p^2(0) + p^2(0)x(0)\right]\right)(\omega t)^2\right.\\
-\left(\frac{5}{6}\frac{p^3(0)}{m\omega} - \frac{5}{12}m\omega\left[x^2(0)p(0) + p(0)x^2(0)\right] - \frac{1}{2}m\omega x(0)p(0)x(0)\right)(\omega t)^3\\
\left.+\left(\frac{11}{24}\left[x(0)p^2(0) + p^2(0)x(0)\right] + \frac{5}{12}p(0)x(0)p(0) - \frac{1}{3}m^2\omega^2 x^3(0)\right)(\omega t)^4\right]
\end{aligned}$$

and

$$
\begin{aligned}
p(t) = p(0)\cos\omega t - m\omega x(0)\sin\omega t \qquad\qquad (3.6)\\
+\beta\left[-\frac{1}{2}m\omega\left[x(0)p^2(0)+p^2(0)x(0)\right](\omega t)\right.\\
-\left(p^3(0)-\frac{1}{4}m^2\omega^2\left[p(0)x^2(0)+x^2(0)p(0)+2x(0)p(0)x(0)\right]\right)(\omega t)^2\\
\left.+\left(\frac{2}{3}m\omega\left[x(0)p^2(0)+p^2(0)x(0)\right]+\frac{1}{2}p(0)x(0)p(0)-\frac{1}{3}m^3\omega^3x^3(0)\right)(\omega t)^3\right]
\end{aligned}
$$

where only terms proportional to first order of β are considered. It is evident that in the limit of $\beta \to 0$ one recover the usual results of ordinary Quantum Mechanics. The term proportional to β shows that the harmonic oscillator is no longer "harmonic" since, now its time evolution is not oscillatory completely. Now for computing expectation values, we need to define physical state.

For completeness we report the expectation value of momentum operator

$$
\begin{aligned}
\frac{\langle\alpha|p(t)|\alpha\rangle}{m} = \frac{p_\alpha(0)}{m}\cos\omega t - \omega x_\alpha(0)\sin\omega t \qquad\qquad (3.7)\\
+\beta\left[-\frac{1}{2}\omega\left[x_\alpha(0)p_\alpha^2(0)+p_\alpha^2(0)x_\alpha(0)\right](\omega t)\right.\\
-\left(\frac{p_\alpha^3(0)}{m}-\frac{1}{4}m\omega^2\left[p_\alpha(0)x_\alpha^2(0)+x_\alpha^2(0)p_\alpha(0)+2x_\alpha(0)p_\alpha(0)x_\alpha(0)\right]\right)(\omega t)^2\\
\left.+\left(\frac{2}{3}\omega\left[x_\alpha(0)p_\alpha^2(0)+p_\alpha^2(0)x_\alpha(0)\right]+\frac{1}{2m}p_\alpha(0)x_\alpha(0)p_\alpha(0)-\frac{1}{3}m^2\omega^3x_\alpha^3(0)\right)(\omega t)^3\right]
\end{aligned}
$$

where $p_\alpha(0) = \langle\alpha|p(0)|\alpha\rangle$ and $x_\alpha(0) = \langle\alpha|x(0)|\alpha\rangle$.

The conclusion is that there is a complicated dependence of the expectation value of the momentum operator with respect to the mass of the oscillator. This is a novel implication which has been induced by the modification of the algebra.

3.1.2 Coherent States

The study of coherent states is important, not only because these states behave like classic oscillators, but also because they represents Gaussian wave packet (states with minimal uncertainty) that do not change shape for all future time.

We shall consider the harmonic oscillator (3.2)

The problem of quantum oscillator is easily solved in terms of annihilation and creation operators a and $a^{\dagger}$. We recall the fundamental definitions (see eq. (1.44))

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{\imath p}{m\omega}\right) \tag{3.8}$$

$$a^{\dagger} = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{\imath p}{m\omega}\right) \tag{3.9}$$

and the inverse relation

$$x = \sqrt{\frac{\hbar}{2m\omega}}\left(a + a^{\dagger}\right) \qquad p = \imath\sqrt{\frac{m\hbar\omega}{2}}\left(-a + a^{\dagger}\right) \tag{3.10}$$

The Hamiltonian H is given in terms of these operators as :

$$H = \hbar\omega\left(a^{\dagger}a + \frac{1}{2}\right) \tag{3.11}$$

If we set $N \equiv a^{\dagger}a$ (Number operator), then

$$\left[N, a^{\dagger}\right] = a^{\dagger}, \quad [N, a] = -a, \quad \left[a^{\dagger}, a\right] = -1 \tag{3.12}$$

Let $\mathbf{H}$ be a Fock space generated by a and $a^{\dagger}$, and $\{|n\rangle\, n \in \{N\}$ $\cup\, \{0\}\}$ be its basis. The action of a and $a^{\dagger}$ on $\mathbf{H}$ are given by

$$a\,|n\rangle = \sqrt{n}\,|n-1\rangle\,, \quad a^{\dagger}\,|n\rangle = \sqrt{n+1}\,|n+1\rangle\,, \quad N\,|n\rangle = n\,|n\rangle \tag{3.13}$$

Where $|0\rangle$ is a normalized vacuum ($a\,|0\rangle = 0 \quad and \quad \langle 0|0\rangle = 1$). Therefore states $|n\rangle$ for $n \geq 1$ are given by

$$|n\rangle = \frac{a^{\dagger n}}{\sqrt{n!}}|0\rangle \tag{3.14}$$

These states satisfy the orthogonality and completeness conditions

$$\langle m|n\rangle = \delta_{mn}, \quad \sum_{n=0}^{\infty}|n\rangle\langle n| = 1 \tag{3.15}$$

By definitions, coherent state is the normalized state $|\lambda\rangle \in H$, which is the eigenstate of annihilation operator and satisfied the following equation,

$$a|\lambda\rangle = |\lambda\rangle \quad \text{where} \quad \langle\lambda|\lambda\rangle = 1 \tag{3.16}$$

and

$$|\lambda\rangle = e^{-\frac{|\lambda|^2}{2}}\sum_{n=0}^{\infty}\frac{\lambda^n}{\sqrt{n!}}|n\rangle = e^{-\frac{|\lambda|^2}{2}}e^{\lambda a^\dagger}|0\rangle \tag{3.17}$$

Actually λ can be complex. The coherent state was introduced by Schrödinger as the quantum state of the harmonic oscillator which minimizes the uncertainty equally distributed in both position x and momentum p.

Let us now discuss a generalization of the previous section where the modification of algebra is taken into account.

3.1.3 Coherent states in modified algebra

Let's consider, for example,

$$a = \frac{1}{\sqrt{2\hbar\omega}}\left(\omega x + \imath\left[p + f(p)\right]\right) \tag{3.18}$$

$$a^\dagger = \frac{1}{\sqrt{2\hbar\omega}}\left(\omega x - \imath\left[p + f(p)\right]\right) \tag{3.19}$$

Here $f(p)$ is a function that satisfies three conditions, namely (in the Fock space) :

1. In the limit $\beta \to 0$ we recover the usual definition for the creation and annihilation operators (3.9);

2. If $\beta \neq 0$, then we have (2.16);

3. $\left[a_{\vec{k}}, a_{\vec{k}'^{\dagger}}\right] = \imath\hbar\delta_{\vec{k}\vec{k}'}$.

It can be shown that the following function satisfies the aforementioned restriction

$$f(p_{\vec{k}}) = \sum_{n=1}^{\infty} \frac{(-\beta)^n}{2n+1} p_{\vec{k}}^{2n+1} \tag{3.20}$$

Condition 3 means that usual results, in relation with the structure of the Fock space, are valid. For instance, the definition of the occupation number operator,($N_{\vec{k}} = a^{\dagger}_{\vec{k}} a_{\vec{k}}$), the interpretation of $a^{\dagger}_{\vec{k}}$ and $a_{\vec{k}}$ are creation and annihilation operators, respectively, etc. Clearly, the relation between $p_{\vec{k}}$, $a_{\vec{k}}$ and $a^{\dagger}_{\vec{k}}$ is not linear, and from the Hamiltonian (3.2) we now deduce that is not diagonal in the occupation number representation. As an example, let assume that $f\left(p_{\vec{k}}\right)$ is of the form

$$f(p_{\vec{k}}) = -\frac{\beta}{3} p_{\vec{k}}^3 \tag{3.21}$$

From (3.18) and (3.19) we find $p_{\vec{k}}$ as a function of $a_{\vec{k}}$ and $a^{\dagger}_{\vec{k}}$, namely

$$p_{\vec{k}} = -\imath\sqrt{\frac{\hbar\omega}{2}}\left(a_{\vec{k}} - a^{\dagger}_{\vec{k}}\right)\left[1 - \sqrt{\frac{\hbar\omega\beta}{8}}\left(a_{\vec{k}} - a^{\dagger}_{\vec{k}}\right)\right] \tag{3.22}$$

While the expression for x remains the usual one (see eq. (3.34) below) It is clear that, if $\beta = 0$ we recover the usual case. Rephrasing the Hamiltonian as a function of creation and annihilation operators we find:

$$H = \sum_{\vec{k}} \hbar\omega\left[N_{\vec{k}} + \sqrt{\frac{\hbar\omega\beta}{8}} g(a_{\vec{k}}, a^{\dagger}_{\vec{k}}) + \beta\frac{(\hbar\omega)^2}{16} h(a_{\vec{k}}, a^{\dagger}_{\vec{k}})\right] \tag{3.23}$$

where the functions $g(a_{\vec{k}}, a^{\dagger}_{\vec{k}})$ and $h(a_{\vec{k}}, a^{\dagger}_{\vec{k}})$ are:

$$g(a_{\vec{k}}, a_{\vec{k}}^{\dagger}) = a_{\vec{k}}^3 - N_{\vec{k}} a_{\vec{k}} - a_{\vec{k}} N_{\vec{k}} - a_{\vec{k}} - (a_{\vec{k}}^{\dagger})^3 + N_{\vec{k}} a_{\vec{k}}^{\dagger} + a_{\vec{k}}^{\dagger} N_{\vec{k}} + a_{\vec{k}}^{\dagger} \tag{3.24}$$

and

$$\begin{aligned} h\left(a_{\vec{k}}, a_{\vec{k}}^{\dagger}\right) &= a_{\vec{k}}^4 + a_{\vec{k}}^2 \left(a_{\vec{k}}^{\dagger}\right)^2 - a_{\vec{k}}^3 a_{\vec{k}}^{\dagger} - a_{\vec{k}}^2 a_{\vec{k}}^{\dagger} a_{\vec{k}} \\ &+ \left(a_{\vec{k}}^{\dagger}\right)^2 a_{\vec{k}}^2 + \left(a_{\vec{k}}^{\dagger}\right)^4 - \left(a_{\vec{k}}^{\dagger}\right)^2 a_{\vec{k}} a_{\vec{k}}^{\dagger} - \left(a_{\vec{k}}^{\dagger}\right)^3 a_{\vec{k}} \\ &- a_{\vec{k}} a_{\vec{k}}^{\dagger} a_{\vec{k}}^2 - a_{\vec{k}} \left(a_{\vec{k}}^{\dagger}\right)^3 + a_{\vec{k}} a_{\vec{k}}^{\dagger} a_{\vec{k}} a_{\vec{k}}^{\dagger} + a_{\vec{k}} \left(a_{\vec{k}}^{\dagger}\right)^2 a_{\vec{k}} \\ &- a_{\vec{k}}^{\dagger} a_{\vec{k}}^3 - a_{\vec{k}}^{\dagger} a_{\vec{k}} \left(a_{\vec{k}}^{\dagger}\right)^2 + a_{\vec{k}}^{\dagger} a_{\vec{k}}^2 a_{\vec{k}}^{\dagger} + a_{\vec{k}}^{\dagger} a_{\vec{k}} a_{\vec{k}}^{\dagger} a_{\vec{k}} \end{aligned} \tag{3.25}$$

Now with these pre- requisites we can consider the coherent states in the context of GUP. Suppose $|\lambda\rangle$ be an eigenstate of the annihilation operator. We remember that the definition of the annihilation operator in GUP may be different from the usual quantum mechanics, while the fact that eigenstates of annihilation operator are coherent states do not change. Therefore one can write

$$a\,|\lambda\rangle = \lambda\,|\lambda\rangle \tag{3.26}$$

Using the eigenstates of the number operator $|n\rangle$ that satisfy completeness and orthogonality conditions, we can expand $|\lambda\rangle$ in terms of the stationary states $|n\rangle$

$$|\lambda\rangle = \sum_{n=0}^{\infty} |n\rangle \langle n|\lambda\rangle = \sum_{n=0}^{\infty} C_n\,|n\rangle \qquad C_n = \langle n|\lambda\rangle \tag{3.27}$$

The eigenvalue equation (3.13) implies the following recursion formula for the expansion coefficients:

$$C_n = \frac{\lambda}{\sqrt{n}} C_{n-1} \tag{3.28}$$

We immediately obtain

$$C_n = \frac{\lambda^n}{\sqrt{n!}} C_0 \tag{3.29}$$

The constant C_0 is determined from the normalization condition on the Fock space,

$$1 = \langle \lambda | \lambda \rangle = |C_0|^2 \sum_{n=0}^{\infty} \frac{\lambda^{2n}}{\sqrt{n!}} = |C_0|^2 e^{|\lambda|^2} \tag{3.30}$$

For any complex number λ the correctly normalized quasi-classical state $|\lambda\rangle$ is therefore given by

$$|\lambda\rangle = e^{-\frac{1}{2}|\lambda|^2} \sum \frac{|\lambda|^n}{\sqrt{n!}} |n\rangle \tag{3.31}$$

We recall that the n-th stationary state $|n\rangle$ is obtained from the ground state wave function by repeated application of the operator $a^\dagger$,

$$|n\rangle = \frac{1}{\sqrt{n!}} \left(a^\dagger\right)^n |0\rangle \tag{3.32}$$

This allows us to write the coherent state in the form:

$$|\lambda\rangle = e^{-\frac{1}{2}|\lambda|^2} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\lambda a^\dagger\right)^n |0\rangle = e^{-\frac{1}{2}|\lambda|^2} e^{\lambda a^\dagger} |0\rangle \tag{3.33}$$

We see that this expression for the eigenstates of the annihilation operator is the same as usual quantum mechanics, equation (3.17). Actually, it is not surprising that there is no change in the form of states by modifying the uncertainty relation as well as for the coherent state. The unchanged state itself cannot be the result of considering generalized uncertainty principle. It is because a quantum state does not necessarily imply a direct connection with the uncertainty principle. Differences caused by different uncertainty relations (such as the GUP) will be found in the expectation values of the operators for a given state and their statistics (such as variance) that can be obtained from the measurement on the state. To analyze the coherent state under the GUP, we should consider $\langle x \rangle$ and $\langle p \rangle$ for the coherent state. This well face in the next subsection.

3.1.4 Coherent states in GUP

Let $|\lambda\rangle$ be a coherent state given by (3.33). In the model under consideration characterized by (3.21) we have the operators x and p that are given by (for a fixed k)

$$x = \sqrt{\frac{\hbar}{2\omega}} \left(a_{\vec{k}} + a^{\dagger}_{\vec{k}} \right) \tag{3.34}$$

and

$$p = -\imath \sqrt{\frac{\hbar\omega}{2}} \left(a_{\vec{k}} - a^{\dagger}_{\vec{k}} \right) \left[1 - \sqrt{\frac{\hbar\omega\beta}{8}} \left(a_{\vec{k}} - a^{\dagger}_{\vec{k}} \right) \right] \tag{3.35}$$

Using (3.34), one finds the following result for the expectation value of position operator x

$$\langle x \rangle = \langle \lambda | x | \lambda \rangle = \sqrt{\frac{\hbar}{2\omega}} \left\langle \lambda | a_{\vec{k}} + a^{\dagger}_{\vec{k}} | \lambda \right\rangle = \sqrt{\frac{\hbar}{2\omega}} \left(\lambda + \lambda^* \right) \tag{3.36}$$

Therefore one has,

$$\langle x \rangle^2 = \frac{\hbar}{2\omega} \left(\lambda^2 + \lambda^{*2} + 2\lambda\lambda^* \right) = \frac{\hbar}{2\omega} \left(\lambda + \lambda^* \right)^2 \tag{3.37}$$

It is straightforward to show that,

$$\left\langle x^2 \right\rangle = \frac{\hbar}{2\omega} \left(\lambda^2 + \lambda^{*2} + 2\lambda\lambda^* + 1 \right) = \frac{\hbar}{2\omega} \left(\lambda + \lambda^* \right)^2 + 1 \tag{3.38}$$

and therefore we find for the variance of x

$$(\triangle x)^2 = \left\langle x^2 \right\rangle - \langle x \rangle^2 = \frac{\hbar}{2\omega} \tag{3.39}$$

This is the same as usual quantum mechanics results. This is not surprising since the definition of position operator is the same as its definition in usual quantum mechanics.

In the same manner, calculations give for the operator p defined in eq. (3.35)

$$\langle p\rangle = -\imath\sqrt{\frac{\hbar\omega}{2}}\left[(\lambda-\lambda^*) - \sqrt{\frac{\hbar\omega\beta}{8}}\left[(\lambda-\lambda^*)^2-1\right]\right] \tag{3.40}$$

and

$$\langle p\rangle^2 = -\frac{\hbar\omega}{2}\left\{(\lambda-\lambda^*)^2 - 2\sqrt{\frac{\hbar\omega\beta}{8}}(\lambda-\lambda^*)\left[(\lambda-\lambda^*)^2-1\right] + \frac{\hbar\omega\beta}{8}\left[(\lambda-\lambda^*)^2-1\right]^2\right\} \tag{3.41}$$

Since,

$$p^2 = -\frac{\hbar\omega}{2}\left[\left(a_{\vec{k}}-a^\dagger_{\vec{k}}\right)^2 - 2\sqrt{\frac{\hbar\omega\beta}{8}}\left(a_{\vec{k}}-a^\dagger_{\vec{k}}\right)^3 + \frac{\hbar\omega\beta}{8}\left(a_{\vec{k}}-a^\dagger_{\vec{k}}\right)^4\right] \tag{3.42}$$

then,

$$\langle p^2\rangle = -\frac{\hbar\omega}{2}\left\{\left[(\lambda-\lambda^*)^2-1\right] - 2\sqrt{\frac{\hbar\omega\beta}{8}}\left(\lambda^3-\lambda^{*3}-3\lambda^*\lambda^2+3\lambda^{*2}\lambda+3\lambda^*-3\lambda\right) + \frac{\hbar\omega\beta}{8}\left(\lambda^4+\lambda^{*4}-4\lambda^*\lambda^3-4\lambda^{*3}\lambda+6\lambda^{*2}\lambda^2-6\lambda^2-6\lambda^{*2}+12\lambda^*\lambda+3\right)\right\} \tag{3.43}$$

and therefore one finds,

$$(\triangle p)^2 = \langle p^2\rangle - \langle p\rangle^2 = -\frac{\hbar\omega}{2}\left[-1 - 2\sqrt{\frac{\hbar\omega\beta}{8}}(2\lambda^*-2\lambda) + \frac{\hbar\omega\beta}{8}\left(-4\lambda^2-4\lambda^{*2}+8\lambda^*\lambda+2\right)\right] \tag{3.44}$$

or by some manipulations,one obtains the following results for the variance p,

$$(\triangle p)^2 = \frac{\hbar\omega}{2} + \hbar\omega\sqrt{\frac{\hbar\omega\beta}{2}}\,(\lambda^* - \lambda) + \frac{\hbar^2\omega^2\beta}{8}\left[1 - 2\,(\lambda^* - \lambda)^2\right] \qquad (3.45)$$

Note that these results give the usual quantum mechanical results when $\beta \to 0$. Equations (3.40) and (3.45) show that although the definition of coherent states does not change in GUP, (because of, for example, deformation of the Heisenberg algebra due to quantum gravitational effect) one can not have coherent state in principle. Now there is a considerable departure from very notion of coherency. In usual quantum mechanics one can have complete coherency in principle. One can localize wave packet in space completely and wave can propagate without broadening. On the contrary in presence of modifications of the canonical commutation relations, one can not localize wave packet at all and it is impossible to omit broadening.

In conclusion, in this chapter, we have obtained the equation of motion for a simple harmonic oscillator in the framework of GUP using Heisenberg picture of quantum mechanics. This situation the terminology of "simple" is no longer applicable because of nonlinear terms in equations of dynamics. It has been shown that there is a complicated mass dependence of momentum operator expectation value which induce a nontrivial departure from usual quantum mechanical results. This is a novel implication of GUP. Moreover we have shown that the definition of coherent states is not different in the framework of GUP with usual quantum mechanics. However there is some considerable difference due to, for example, gravitational uncertainty. Because of these uncertainties, in quantum gravity one can not have solitonic states and any wave packet will be broad. The main results of our calculations, is that in the framework of GUP the notion of coherency breaks down [15].

Conclusion

All fundamental theories of unification, such as: String theory, loop quantum gravity, quantum gravity predict the existence of a minimal length of the order of Planck length scale. The model which are designed to implement the minimal length scale have been analysed enter the Generalized Uncertainty Principle (GUP).

In this thesis GUP has been studied in different frameworks and has been shown how canonical commutation relation are modified.

In the end, Harmonic Oscillator has been studied with the result that the oscillation is not longer harmonic because of the presence of nonlinear terms. It has been shown, in fact, that there is a mass dependence of momentum operator expectation value (3.1.1). It has been shown that both in standard Quantum Mechanics and in the modified one, the coherent states do not change, although there is a β-correction in the canonical commutation relations. The main result of this analysis is that the uncertainty principle for coherent states turns out modified in the framework of GUP, i.e. the β terms in canonical commutation relations are taken into account.

Appendix A

Hilbert Space Representation

In this appendix we will construct a new *Hilbert space representation* which is compatible with our commutation relation in GUP. Fortunately, by neglecting the presence of a minimal uncertainty in momentum, there still would exist a continuous momentum space representation, which means that we can explore the physical implications of the minimal length by working with the convenient representation of the commutation relations on momentum space wave functions.

A.1 Some consequences on momentum space

In this subsection we consider the momentum space representation. To obtain a minimum measurable uncertainty, the inequality (2.19) on the boundary of allowed region gives

$$\triangle x \triangle p = \frac{\hbar}{2}\left(1 - 2\alpha \langle p \rangle + 4\alpha^2 \langle p^2 \rangle\right) \tag{A.1}$$

Using $\langle p^2 \rangle = (\triangle p)^2 + \langle p \rangle^2$, this relation can be rewritten as a second order equation for $\triangle p$. The solution for $\triangle p$ are

$$\triangle p = \frac{\triangle x}{4\alpha^2 \hbar} \pm \sqrt{\left(\frac{\triangle x}{4\alpha^2 \hbar}\right)^2 - \frac{\langle p \rangle}{2\alpha}\left(2\alpha \langle p \rangle - 1\right) - \frac{1}{4\alpha^2}} \tag{A.2}$$

The reality of solutions gives the following minimum value for $\triangle x$

$$\triangle x_{min}\left(\langle p\rangle\right) = 2\alpha\hbar\sqrt{1-2\alpha\left\langle p\right\rangle + 4\alpha^2\left\langle p\right\rangle^2} \tag{A.3}$$

therefore the absolutely smallest uncertainty in position, where $\langle p\rangle = 0$, would be

$$\triangle x_0 = 2\alpha\hbar \tag{A.4}$$

Now, in our momentum space, we take operators $\mathbf{P}$ and $\mathbf{X}$ in the form

$$\mathbf{P} = p \tag{A.5}$$

$$\mathbf{X} = \left(1-\alpha p+2\alpha^2p^2\right)x \tag{A.6}$$

where $x = \imath\hbar\frac{\partial}{\partial p}$. Then by operating on momentum space wave function, we have

$$\mathbf{P}\varphi\left(p\right) = p\varphi\left(p\right) \tag{A.7}$$

$$\mathbf{X}\varphi\left(p\right) = \imath\hbar\left(1-\alpha p+2\alpha^2p^2\right)\frac{\partial}{\partial p}\varphi\left(p\right) \tag{A.8}$$

The scalar product in this representation should be modified due to the presence of the additional factor $(\mathbf{1}-\alpha\mathbf{p}+\mathbf{2}\alpha^{\mathbf{2}}\mathbf{p}^{\mathbf{2}}) \equiv \mathbf{G_{Mm}}\left(\mathbf{p}\right)$ and the existence of a maximal momentum as

$$\langle\Phi|\varphi\rangle = \int_{-P_{pl}}^{+P_{pl}}\frac{dp}{1-\alpha p+2\alpha^2p^2}\Phi^*\left(p\right)\varphi\left(p\right) \tag{A.9}$$

Here, the presence of the term $-\alpha p$ in $G_{Mm}\left(p\right)$ implies the existence of a maximal particle's momentum (the Planck momentum, $P_{pl} \equiv M_{pl}c$) which affects the scalar product as we see in eq. (A.9). In this framework, the identity operator would be represented as

$$\mathbf{1} = \int_{-P_{pl}}^{+P_{pl}} \frac{dp}{1 - \alpha p + 2\alpha^2 p^2} \left|p\right\rangle \left\langle p\right| \tag{A.10}$$

and the scalar product of the momentum eigenstate changes to

$$\left\langle p|p^{'}\right\rangle = \left(1 - \alpha p + 2\alpha^2 p^2\right) \delta\left(p - p^{'}\right) \tag{A.11}$$

The existence of maximal particle's momentum in addition to minimal observable length, has several nwe implication on the Hilbert space representation.

Appendix B

Baker-Hausdorff lemma

In mathematics, the Baker–Campbell–Hausdorff formula is the solution to the equation

$$Z = \log\left(e^X e^Y\right) \tag{B.1}$$

for possibly noncommutative X and Y in the Lie algebra of a Lie group. This formula tightly links Lie groups to Lie algebras by expressing the logarithm of the product of two Lie group elements as a Lie algebra element using only Lie algebraic operations.

The solution of the equation (B.1), interrupted at third order is given by

$$Z = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}([X,[X,Y]] - [Y,[X,Y]]) + \dots \tag{B.2}$$

As we can see, in the case of commutative Lie algebra, the formula reduces to the usual sum of the two operators.

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