Fourier Transform of a Unit Step Function

The unit *step function* is

$$u(t) = \begin{cases} 1 \text{ if } u \geq 0 \\ 0 \text{ if } u < 0 \end{cases}$$

and the *signum function* is

$$\operatorname{sgn}(t) = \begin{cases} 1 \text{ if } u \geq 0 \\ -1 \text{ if } u < 0. \end{cases}$$

These two functions are related by the equation

$$u(t) = \frac{1 + \operatorname{sgn}(t)}{2}.$$

Theorem 0.1. *The Fourier transform of sgn*(t) *is* $F(\omega) = \frac{2}{j\omega}$.

Proof. Note that (with $a > 0$):

$$\operatorname{sgn}(t) = \lim_{a \to 0} \left(e^{-at} u(t) - e^{at} u(-t) \right).$$

Thus,

$$\begin{aligned}
\mathcal{F}(\operatorname{sgn}(t)) &= \lim_{a \to 0} \left(\mathcal{F}\left(e^{-at} u(t)\right) - \mathcal{F}\left(e^{at} u(-t)\right) \right) \\
&= \lim_{a \to 0} \left(\int_0^{\infty} e^{-at} e^{-j\omega t} dt - \int_{-\infty}^{0} e^{at} e^{-j\omega t} dt \right) \\
&= \lim_{a \to 0} \left(\frac{-1}{a + j\omega} \cdot e^{-(a+j\omega)t} \Big|_0^{\infty} - \frac{1}{a - j\omega} \cdot e^{(a-j\omega)t} \Big|_{-\infty}^{0} \right) \\
&= \lim_{a \to 0} \left(\frac{-1}{a + j\omega} \cdot (0 - 1) - \frac{1}{a - j\omega} \cdot (1 - 0) \right) \\
&= \lim_{a \to 0} \left(\frac{1}{a + j\omega} - \frac{1}{a - j\omega} \right) \\
&= \lim_{a \to 0} \frac{-2j\omega}{a^2 + \omega^2} \\
&= \frac{2}{j\omega}
\end{aligned}$$

■

Theorem 0.2. *The Fourier transform of* $u(t)$ *is* $F(\omega) = \frac{1}{j\omega} + \pi\delta(\omega)$.

Proof. The theorem above shows that

$$\operatorname{sgn}(t) \longleftrightarrow \frac{2}{j\omega}.$$

We also know that

$$1 \longleftrightarrow 2\pi\delta(\omega).$$

Thus,

$$\begin{aligned}
\mathcal{F}\left(u(t)\right) &= \mathcal{F}\left(\frac{1}{2}\left(1 + \mathrm{sgn}(t)\right)\right) \\
&= \frac{1}{2}\mathcal{F}\left(1\right) + \frac{1}{2}\mathcal{F}\left(\mathrm{sgn}(t)\right) \\
&= \frac{1}{2}(\pi\delta(\omega)) + \frac{1}{2}\left(\frac{2}{j\omega}\right) \\
&= \pi\delta(\omega) + \frac{1}{j\omega}
\end{aligned}$$

■