UNIT V

Curve Fitting and Solution of Equation

5.1 CURVE FITTING

In many branches of applied mathematics and engineering sciences we come across experiments and problems, which involve two variables. For example, it is known that the speed v of a ship varies with the horsepower p of an engine according to the formula $p = a + bv^3$. Here a and b are the constants to be determined. For this purpose we take several sets of readings of speeds and the corresponding horsepowers. The problem is to find the best values for a and b using the observed values of v and p. Thus, the general problem is to find a suitable relation or law that may exist between the variables x and y from a given set of observed values (x_i, y_i), $i = 1, 2, \ldots\ldots\ldots, n$. Such a relation connecting x and y is known as empirical law. For above example, $x = v$ and $y = p$.

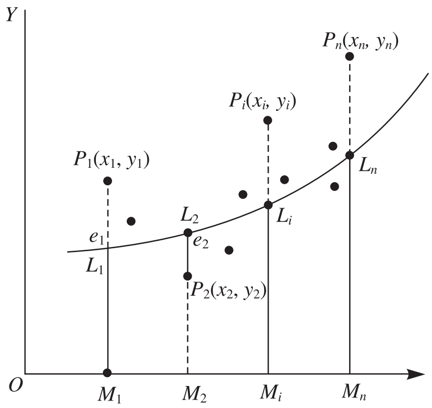
The process of finding the equation of the curve of best fit, which may be most suitable for predicting the unknown values, is known as curve fitting. Therefore, curve fitting means an exact relationship between two variables by algebraic equations. There are following methods for fitting a curve.

I. Graphic method
II. Method of group averages
III. Method of moments
IV. Principle of least square.

Out of above four methods, we will only discuss and study here principle of least square.

5.2 PRINCIPLE OF LEAST SQUARES

The graphical method has the drawback in that the straight line drawn may not be unique but principle of least squares provides a unique set of values to the constants and hence suggests a curve of best fit to the given data. The method of least square is probably the most systematic procedure to fit a unique curve through the given data points.



Let the curve $y = a + bx + cx^2 + ...kx^{m-1}$ ...(1)

be fitted to the set of n data points (x_1, y_1), (x_2, y_2), (x_3, y_3),......, (x_n, y_n). At $(x = x_i)$ the observed (or experimental) value of the ordinate is $y_i = P_iM_i$ and the corresponding value on the fitting curve (i) is $a + bx_i + cx_i^2 + ...kx_i^m = L_iM_i$ which is the expected or calculated value. The difference of the observed and the expected value is $P_iM_i - L_iM_i = e_i$(say) this difference is called error at $(x = x_i)$ clearly some of the error $e_1, e_2, e_3, ..., e_i, ..., e_n$ will be positive and other negative. To make all errors positive we square each of the errors *i.e.* $S = e_1^2 + e_2^2 + e_3^2 + ... + e_i^2 + ... + e_n^2$ the curve of best fit is that for which e's are as small as possible *i.e.* S, the sum of the square of the errors is a minimum this is known as the principle of least square. The theoretical values for $x_1, x_2, ..., x_n$ may be $y_{\lambda_1}, y_{\lambda_2}, ...y_{\lambda_n}$.

5.3 FITTING OF STRAIGHT LINE

Let a straight line $y = a + bx$ ...(1)

which is fitted to the given data points (x_1, y_1), (x_2, y_2), (x_3, y_3),......, (x_n, y_n).

Let y_{λ_1} be the theoretical value for x_1 then $e_1 = y_1 - y_{\lambda_1}$

$\Rightarrow$ $$e_1 = y_1 - (a + bx_1)$$

$\Rightarrow$ $$e_1^2 = (y_1 - a - bx_1)^2$$

Now we have $$S = e_1^2 + e_2^2 + e_3^2 + ... + e_n^2$$

$$S = \sum_{i=1}^{n} e_i^2$$

$$S = \sum_{i=1}^{n} (y_i - a - bx_i)^2$$

By the principle of least squares, the value of S is minimum therefore,

$$\frac{\partial S}{\partial a} = 0 \quad ...(2)$$

and $$\frac{\partial S}{\partial b} = 0 \quad \ldots(3)$$

On solving equations (2) and (3), and dropping the suffix, we have

$$\sum y = na + b\sum x \quad \ldots(4)$$

$$\sum xy = a\sum x + b\sum x^2 \quad \ldots(5)$$

The equation (3) and (4) are known as normal equations.

On solving equations (3) and (4), we get the value of a and b. Putting the value of a and b in equation (1), we get the equation of the line of best fit.

5.4 FITTING OF PARABOLA

Let a parabola $y = a + bx + cx^2$ ...(1)

which is fitted to a given data (x_1, y_1), (x_2, y_2), (x_3, y_3) ,……, (x_n, y_n) .

Let y_λ be the theoretical value for x_1 then $e_1 = y_1 - y_\lambda$

$$\Rightarrow \quad e_1 = y_1 - (a + bx_1 + cx_1^2)$$

$$\Rightarrow \quad e_1^2 = (y_1 - a - bx_1 - cx_1^2)^2$$

Now we have $$S = \sum_{i=1}^{n} e_i^2$$

$$S = \sum_{i=1}^{n} (y_i - a - bx_i - cx_i^2)^2$$

By the principle of least squares, the value of S is minimum, therefore

$$\frac{\partial S}{\partial a} = 0, \frac{\partial S}{\partial b} = 0 \text{ and } \frac{\partial S}{\partial c} = 0 \quad \ldots(2)$$

Solving equation (2) and dropping suffix, we have

$$\sum y = na + b\sum x + c\sum x^2 \quad \ldots(3)$$

$$\sum xy = a\sum x + b\sum x^2 + c\sum x^3 \quad \ldots(4)$$

$$\sum x^2 y = a\sum x^2 + b\sum x^3 + c\sum x^4 \quad \ldots(5)$$

The equation (3), (4) and (5) are known as normal equations.

On solving equations (3), (4) and (5), we get the values of *a,b* and *c*. Putting the values of *a, b* and *c* in equation (1), we get the equation of the parabola of best fit.

5.5 CHANGE OF SCALE

When the magnitude of the variable in the given data is large number then calculation becomes very much tedious then problem is further simplified by taking suitable scale when the value of x are given at equally spaced intervals.

Let h be the width of the interval at which the values of x are given and let the origin of x and y be taken at the point x_0, y_0 respectively, then putting

$$u = \frac{(x - x_0)}{h} \text{ and } v = y - y_0$$

If m is odd then, $u = \dfrac{x - (\text{middle term})}{\text{interval}(h)}$

But if m is even then, $u = \dfrac{x - (\text{middle of two middle term})}{\frac{1}{2}\ (\text{interval})}$

Example 1: Find the best-fit values of a and b so that $y = a + bx$ fits the data given in the table.

x :	0	1	2	3	4
y :	1	1.8	3.3	4.5	6.3

Sol. Let the straight line is $y = a + bx$ ...(1)

x	y	xy	x^2
0	1	0	0
1	1.8	1.8	1
2	3.3	6.6	4
3	4.5	13.5	9
4	6.3	25.2	16
$\sum x = 10$	$\sum y = 16.9$	$\sum xy = 47.1$	$\sum x^2 = 30$

Normal equations are, $\sum y = na + b\sum x$ ...(2)

$$\sum xy = a\sum x + b\sum x^2 \quad ...(3)$$

Here $n = 5$, $\sum x = 10$, $\sum y = 16.9$, $\sum xy = 47.1$, $\sum x^2 = 30$

Putting these values in normal equations, we get

$$16.9 = 5a + 10b$$

$$47.1 = 10a + 30b$$

On solving these two equations, we get

$$a = 0.72\ , \quad b = 1.33.$$

So required line $y = 0.72 + 1.33x$. **Ans.**

Example 2: Fit a straight line to the given data regarding x as the independent variable.

x	1	2	3	4	5	6
y	1200	900	600	200	110	50

Sol. Let the straight line obtained from the given data by $y = a + bx$ ...(1)

Then the normal equations are $\sum y = na + b\sum x$ ...(2)

$\sum xy = a\sum x + b\sum x^2$ ...(3)

x	y	x^2	xy
1	1200	1	1200
2	900	4	1800
3	600	9	1800
4	200	16	800
5	110	25	550
6	50	36	300
$\sum x = 21$	$\sum y = 3060$	$\sum x^2 = 91$	$\sum xy = 6450$

Putting all values in the equations (2) and (3), we get

$$3060 = 6a + 21b$$

$$6450 = 21a + 91b$$

Solving these equations, we get

$$a = 1361.97 \quad \text{and} \quad b = -243.42$$

Hence the fitted equation is $y = 1361.97 - 243.42x$. **Ans.**

Example 3: Fit a straight line to the following data:

x	71	68	73	69	67	65	66	67
y	69	72	70	70	68	67	68	64

Sol. Here we from the following table:

x	y	xy	x^2
71	69	4899	5041
68	72	4896	4624
73	70	5110	5329
69	70	4830	4761
67	68	4556	4489
65	67	4355	4225
66	68	4488	4356
67	64	4288	4489
$\Sigma x = 546$	$\Sigma y = 548$	$\Sigma xy = 37422$	$\Sigma x^2 = 37314$

Let the equation of straight line to be fitted be

$$y = a + bx \quad ...(1)$$

And the normal equations are

$$\Sigma y = an + b\Sigma x \quad ...(2)$$

$$\Sigma xy = a\Sigma x + b\Sigma x^2 \quad ...(3)$$

$\Rightarrow$

$$8a + 546b = 548$$

$$546a + 37314b = 37422$$

Solving these equations, we get

$$a = 39.5454,\ b = 0.4242$$

Hence from (1)

$$y = 39.5454 + 0.4242x. \quad \textbf{Ans.}$$

Example 4: Find the least square polynomial approximation of degree two to the data.

x	0	1	2	3	4
y	–4	–1	4	11	20

Also compute the least error.

Sol. Let the equation of the polynomial be $y = a + bx + cx^2$ ...(1)

x	y	xy	x^2	x^2y	x^3	x^4
0	–4	0	0	0	0	0
1	–1	–1	1	–1	1	1
2	4	8	4	16	8	16
3	11	33	9	99	27	81
4	20	80	16	320	64	256
$\sum x = 10$	$\sum y = 30$	$\sum xy = 120$	$\sum x^2 = 30$	$\sum x^2y = 434$	$\sum x^3 = 100$	$\sum x^4 = 354$

The normal equations are,

$$\sum y = na + b\sum x + c\sum x^2 \quad ...(2)$$

$$\sum xy = a\sum x + b\sum x^2 + c\sum x^3 \quad ...(3)$$

$$\sum x^2y = a\sum x^2 + b\sum x^3 + c\sum x^4 \quad ...(4)$$

Here $n = 5$, $\sum x = 10$, $\sum y = 30$, $\sum xy = 120$, $\sum x^2 = 30$, $\sum x^2y = 434$, $\sum x^3 = 100$, $\sum x^4 = 354$.

Putting all these values in (2), (3) and (4), we get

$$30 = 5a + 10b + 30c \quad ...(5)$$

$$120 = 10a + 30b + 100c \quad ...(6)$$

$$434 = 30a + 100b + 354c \quad ...(7)$$

On solving these equations, we get $a = -4$, $b = 2$, $c = 1$. Therefore required polynomial is $y = -4 + 2x + x^2$, errors = 0. **Ans.**

Example 5: Fit a second degree curve of regression of y on x to the following data:

x	1	2	3	4
y	6	11	18	27

Sol. We form the following table:

x	y	x^2	x^3	x^4	xy	x^2y
1	6	1	1	1	6	6
2	11	4	8	16	22	44
3	18	9	27	81	54	162
4	27	16	64	256	108	432
$\Sigma x = 10$	$\Sigma y = 62$	$\Sigma x^2 = 30$	$\Sigma x^3 = 100$	$\Sigma x^4 = 354$	$\Sigma xy = 190$	$\Sigma x^2y = 644$

The equation of second degree parabola is given by

$$y = a + bx + cx^2 \quad \ldots(1)$$

And the normal equations are

$$\Sigma y = an + b\Sigma x + c\Sigma x^2 \quad \ldots(2)$$

$$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3 \quad \ldots(3)$$

$$\Sigma x^2 y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4 \quad \ldots(4)$$

$$\Rightarrow \quad \left.\begin{aligned} 4a + 10b + 30c &= 62 \\ 10a + 30b + 100c &= 190 \\ 30a + 100b + 354c &= 644 \end{aligned}\right\} \Rightarrow a = 3, b = 2, c = 1$$

Hence $y = 3 + 2x + x^2$. **Ans.**

Example 6: By the method of least squares, find the straight line that best fits the following data:

x	1	2	3	4	5	6	7
y	14	27	40	55	68	77	85

Sol. The equation of line is

$$y = a + bx \quad \ldots(1)$$

The normal equations are $\Sigma y = an + b\Sigma x$ ...(2)

and $\Sigma xy = a\Sigma x + b\Sigma x^2$ ...(3)

Now we from the following table:

x	y	xy	x^2
1	14	14	1
2	27	54	4
3	40	120	9
4	55	220	16
5	68	340	25
6	77	462	36
7	85	595	49
$\Sigma x = 28$	$\Sigma y = 356$	$\Sigma xy = 1805$	$\Sigma x^2 = 140$

∴ From equations (2) and (3), we get

$$7a + 28b = 356$$

$$28a + 140b = 1805$$

On solving these equations, we get

$$a = -3.5714$$

$$b = 13.6071$$

∴ $y = -3.5714 + 13.6071x$. **Ans.**

Example 7: Find the least squares fit of the form $y = a_0 + a_1 x^2$ to the following data

x	−1	0	1	2
y	2	5	3	0

Sol. We have $y = a_0 + a_1 x^2$

By principle of least squares

$$s = \sum_i \left\{ y_i - (a_0 + a_1 x_i^2 \right\}^2$$

$$\frac{\partial s}{\partial a} = \sum_i 2\left\{ y_i - (a_0 + a_1 x_i^2 \right\}(-x_i^2) = 0$$

⇒ $$\Sigma x^2 y = a_0 \Sigma x^2 + a_1 \Sigma x^4 \quad \text{(Drop suffix)} \qquad \ldots(1)$$

and $$\frac{\partial s}{\partial a_0} = \sum -2\left\{y_i - (a_0 + a_1 x_i^2)\right\} = 0$$

$\Rightarrow$ $$\Sigma y = a_0 n + a_1 \Sigma x^2 \quad \text{(Drop suffix)} \qquad ...(2)$$

Now, we form the following table:

x	y	x^2	x^2y	x^4
–1	2	1	2	1
0	5	0	0	0
1	3	1	3	1
2	0	4	0	16
$\Sigma x = 2$	$\Sigma y = 10$	$\Sigma x^2 = 6$	$\Sigma x^2 y = 5$	$\Sigma x^4 = 18$

From equations (1) and (2), we get

$$6a_0 + 18a_1 = 5 \qquad ...(3)$$

and $$4a_0 + 6a_1 = 10 \qquad ...(4)$$

Solving the equations (3) and (4), we get

$$a_1 = -\ 1.111,\ a_0 = 4.166$$

$\therefore$ The equation is given by

$$y = 4.166 - 1.111x^2.$$ **Ans.**

Example 8: Fit a second-degree parabola to the following data taking x as the independent variable.

x	1	2	3	4	5	6	7	8	9
y	2	6	7	8	10	11	11	10	9

Sol. The equation of second-degree parabola is given by $y = a + bx + cx^2$ and the normal equations are:

$$\left.\begin{aligned} \sum y &= na + b\sum x + c\sum x^2 \\ \sum xy &= a\sum x + b\sum x^2 + c\sum x^3 \\ \sum x^2 y &= a\sum x^2 + b\sum x^3 + c\sum x^4 \end{aligned}\right\} \qquad ...(1)$$

Here $n = 9$. The various sums are appearing in the table as follows:

x	y	xy	x^2	x^2y	x^3	x^4
1	2	2	1	2	1	1
2	6	12	4	24	8	16
3	7	21	9	63	27	81
4	8	32	16	128	64	256
5	10	50	25	250	125	625
6	11	66	36	396	216	1296
7	11	77	49	539	343	2401
8	10	80	64	640	512	4096
9	9	81	81	729	729	6561
$\sum x = 45$	$\sum y = 74$	$\sum xy = 421$	$\sum x^2 = 284$	$\sum x^2y = 2771$	$\sum x^3 = 2025$	$\sum x^4 = 15333$

Putting these values of $\sum x, \sum y, \sum x^2, \sum xy, \sum x^2 y, \sum x^3$ and $\sum x^4$ in equation (1) and solving the equations for a, b and c, we get

$$a = -0.923\,;\; b = 3.520\,;\; c = -0.267\cdot$$

Hence the fitted equation is

$$y = -0.923 + 3.53x - 0.267x^2.$$ **Ans.**

Example 9: Show that the line of fit to the following data is given by $y = 0.7x + 11.28$.

x	0	5	10	15	20	25
y	12	15	17	22	24	30

Sol. Here $n = 6$ (even)

Let $x_0 = 12.5,\ h = 5,\ y_0 = 20$ (say)

Then, $u = \dfrac{x - 12.5}{2.5}$ and $v = y - 20$, we get

x	y	u	v	uv	u^2
0	12	–5	–8	40	25
5	15	–3	–5	15	9
10	17	–1	–3	3	1
15	22	1	2	2	1
20	24	3	4	12	9
25	30	5	10	50	25
		$\sum u = 0$	$\sum v = 0$	$\sum uv = 122$	$\sum u^2 = 70$

The normal equations are,

$$0 = 6a + 0\cdot b \quad \Rightarrow \quad a = 0$$

$$122 = 0\cdot a + 70b \quad \Rightarrow \quad b = 1.743$$

Thus line of fit is $v = 1.743u$.

or $$y - 20 = (1.743)\left(\frac{x - 12.50}{2.5}\right) = 0.69x - 8.715$$

or $$y = 0.7x + 11.285. \quad \textbf{Ans.}$$

Example 10: Fit a second-degree parabola to the following data by least squares method.

x	1929	1930	1931	1932	1933	1934	1935	1936	1937
y	352	356	357	358	360	361	361	360	359

Sol. Taking $x_0 = 1933$, $y_0 = 357$ then $u = \frac{(x - x_0)}{h}$

Here $h = 1$

Taking $u = x - x_0$ and $v = y - y_0$, therefore, $u = x - 1933$ and $v = y - 357$

x	$u = x - 1933$	y	$v = y - 357$	uv	u^2	u^2v	u^3	u^4
1929	–4	352	–5	20	16	–80	–64	256
1930	–3	356	–1	3	9	–9	–27	81
1931	–2	357	0	0	4	0	–8	16
1932	–1	358	1	–1	1	1	–1	1
1933	0	360	3	0	0	0	0	0
1934	1	361	4	4	1	4	1	1
1935	2	361	4	8	4	16	8	16
1936	3	360	3	9	9	27	27	81
1937	4	359	2	8	16	32	64	256
Total	$\sum u = 0$		$\sum v = 11$	$\sum uv = 51$	$\sum u^2 = 60$	$\sum u^2v = -9$	$\sum u^3 = 0$	$\sum u^4 = 708$

Then the equation $y = a + bx + cx^2$ is transformed to $v = A + Bu + Cu^2$ ...(1)

Normal equations are:

$$\sum v = 9A + B\sum u + C\sum u^2 \quad \Rightarrow 11 = 9A + 60C$$
$$\sum uv = A\sum u + B\sum u^2 + C\sum u^3 \quad \Rightarrow B = 17/20$$
$$\sum u^2v = A\sum u^2 + B\sum u^3 + C\sum u^4 \quad \Rightarrow -9 = 60A + 708C$$

On solving these equations, we get $A = \frac{694}{231} = 3$, $B = \frac{17}{20} = 0.85$ and $C = -\frac{247}{924} = -0.27$

$\therefore$ $$v = 3 + 0.85u - 0.27u^2$$

$\Rightarrow$ $$y - 357 = 3 + 0.85(x - 1933) - 0.27(x - 1933)^2$$

$\Rightarrow$ $$y = -1010135.08 + 1044.69x - 0.27x^2. \quad \textbf{Ans.}$$

Example 11: Fit second degree parabola to the following:

x	0	1	2	3	4
y	1	1.8	1.3	2.5	6.3

Sol. Here $n = 5$ (odd) therefore $x_0 = 2$, $h = 1$, $y_0 = 0$ (say)

Now let $u = x - 2$, $v = y$ and the curve of fit be $v = a + bu + cu^2$.

x	y	u	v	uv	u^2	u^2v	u^3	u^4
0	1	–2	1	–2	4	4	–8	16
1	1.8	–1	1.8	–1.8	1	1.8	–1	1
2	1.3	0	1.3	0	0	0	0	0
3	2.5	1	2.5	2.5	1	2.5	1	1
4	6.3	2	6.3	12.6	4	25.2	8	16
Total		0	12.9	11.3	10	33.5	0	34

Hence the normal equations are,

$$\sum v = 5a + b\sum u + c\sum u^2$$

$$\sum uv = a\sum u + b\sum u^2 + c\sum u^3$$

$$\sum u^2 v = a\sum u^2 + b\sum u^3 + c\sum u^4$$

On putting the values of $\sum u, \sum v$ etc. from the table in these, we get

$$12.9 = 5a + 10c,\ 11.3 = 10b,\ 33.5 = 10a + 34c.$$

On solving these equations, we get

$$a = 1.48,\ b = 1.13 \text{ and } c = 0.55$$

Therefore the required equation is $v = 1.48 + 1.13u + 0.55u^2$.

Again substituting $u = x - 2$ and $v = y$, we get

$$y = 1.48 + 1.13(x - 2) + 0.55(x - 2)^2$$

or $$y = 1.42 - 1.07x + 0.55x^2.$$ **Ans.**

5.6 FITTING OF AN EXPONENTIAL CURVE

Suppose an exponential curve of the form

$$y = ae^{bx}$$

Taking logarithm on both the sides, we get

$$\log_{10} y = \log_{10} a + bx \log_{10} e$$

i.e., $$Y = A + Bx \qquad \ldots(1)$$

where $$Y = \log_{10} y,\ A = \log_{10} a \text{ and } B = b\log_{10} e.$$

The normal equations for (1) are,

$$\sum Y = nA + B\sum x$$

$$\sum xY = A\sum x + B\sum x^2$$

On solving the above two equations, we get A and B

then $$a = \text{antilog } A,\ b = \frac{B}{\log_{10} e}$$

5.7 FITTING OF THE CURVE $y = ax + bx^2$

Error of estimate for ith point (x_i, y_i) is

$$e_i = (y_i - ax_i - bx_i^2)$$

We have, $$S = \sum_{i=1}^{n} e_i^2$$

$$= \sum_{i=1}^{n} (y_i - ax_i - bx_i^2)^2$$

By the principle of least square, the value of S is minimum

$\therefore$ $$\frac{\partial S}{\partial a} = 0 \text{ and } \frac{\partial S}{\partial b} = 0$$

Now $$\frac{\partial S}{\partial a} = 0$$

$\Rightarrow$ $$\sum_{i=1}^{n} 2(y_i - ax_i - bx_i^2)(-x_i) = 0$$

or $$\sum_{i=1}^{n} x_i y_i = a\sum_{i=1}^{n} x_i^2 + b\sum_{i=1}^{n} x_i^3 \quad ...(1)$$

and $$\frac{\partial S}{\partial b} = 0$$

$\Rightarrow$ $$\sum_{i=1}^{n} 2(y_i - ax_i - bx_i^2)(-x_i^2) = 0$$

or $$\sum_{i=1}^{n} x_i^2 y_i = a\sum_{i=1}^{n} x_i^3 + b\sum_{i=1}^{n} x_i^4 \quad ...(2)$$

Dropping the suffix i from (1) and (2), then the normal equations are,

$$\sum xy = a\sum x^2 + b\sum x^3$$

$$\sum x^2 y = a\sum x^3 + b\sum x^4$$

5.8 FITTING OF THE CURVE $y = ax + \frac{b}{x}$

Error of estimate for ith point (x_i, y_i) is

$$e_i = (y_i - ax_i - \frac{b}{x_i})$$

We have,
$$S = \sum_{i=1}^{n} e_i^2$$
$$= \sum_{i=1}^{n} (y_i - ax_i - \frac{b}{x_i})^2$$

By the principle of least square, the value of S is minimum

$$\therefore \quad \frac{\partial S}{\partial a} = 0 \text{ and } \frac{\partial S}{\partial b} = 0$$

Now
$$\frac{\partial S}{\partial a} = 0$$

$$\Rightarrow \quad \sum_{i=1}^{n} 2(y_i - ax_i - \frac{b}{x_i})(-x_i) = 0$$

or
$$\sum_{i=1}^{n} x_i y_i = a\sum_{i=1}^{n} x_i^2 + nb \qquad \ldots(1)$$

and
$$\frac{\partial S}{\partial b} = 0$$

$$\Rightarrow \quad \sum_{i=1}^{n} 2(y_i - ax_i - \frac{b}{x_i})(-\frac{1}{x_i}) = 0$$

or
$$\sum_{i=1}^{n} \frac{y_i}{x_i} = na + b\sum_{i=1}^{n} \frac{1}{x_i^2} \qquad \ldots(2)$$

Dropping the suffix i from (1) and (2), then the normal equations are,

$$\sum xy = nb + a\sum x^2$$

$$\sum \frac{y}{x} = na + b\sum \frac{1}{x^2}$$

where n is the number of pair of values of x and y.

5.9 FITTING OF THE CURVE $y = \frac{c_0}{x} + c_1\sqrt{x}$

Error of estimate for ith point (x_i, y_i) is

$$e_i = (y_i - \frac{c_0}{x_i} - c_1\sqrt{x_i})$$

We have,
$$S = \sum_{i=1}^{n} e_i^2$$
$$= \sum_{i=1}^{n} (y_i - \frac{c_0}{x_i} - c_1\sqrt{x_i})^2$$

By the principle of least square, the value of S is minimum

$$\therefore \quad \frac{\partial S}{\partial c_0} = 0 \text{ and } \frac{\partial S}{\partial c_1} = 0$$

Now
$$\frac{\partial S}{\partial c_0} = 0$$

$$\Rightarrow \quad \sum_{i=1}^{n} 2(y_i - \frac{c_0}{x_i} - c_1\sqrt{x_i})(-\frac{1}{x_i}) = 0$$

or
$$\sum_{i=1}^{n} \frac{y_i}{x_i} = c_0 \sum_{i=1}^{n} \frac{1}{x_i^2} + c_1 \sum_{i=1}^{n} \frac{1}{\sqrt{x_i}} \qquad \ldots(1)$$

and
$$\frac{\partial S}{\partial c_1} = 0$$

$$\Rightarrow \quad \sum_{i=1}^{n} 2(y_i - \frac{c_0}{x_i} - c_1\sqrt{x_i})(-\sqrt{x_i}) = 0$$

or
$$\sum_{i=1}^{n} y_i\sqrt{x_i} = c_0 \sum_{i=1}^{n} \frac{1}{\sqrt{x_i}} + c_1 \sum_{i=1}^{n} x_i \qquad \ldots(2)$$

Dropping the suffix i from (1) and (2), then the normal equations are,

$$\sum \frac{y}{x} = c_0 \sum \frac{1}{x^2} + c_1 \sum \frac{1}{\sqrt{x}}$$

$$\sum y\sqrt{x} = c_0 \sum \frac{1}{\sqrt{x}} + c_1 \sum x.$$

Example 12: Find the curve of best fit of the type $y = ae^{bx}$ to the following data by the method of least squares:

x	1	5	7	9	12
y	10	15	12	15	21

Sol. The curve to be fitted is $y = ae^{bx}$ or $Y = A + Bx$, where $Y = \log_{10} y$, $A = \log_{10} a$, and $B = b \log_{10} e$.

Therefore the normal equations are:

$$\sum Y = 5A + B\sum x$$

$$\sum xY = A\sum x + B\sum x^2$$

x	y	$Y = \log_{10} y$	x^2	xY
1	10	1.0000	1	1
5	15	1.1761	25	5.8805
7	12	1.0792	49	7.5544
9	15	1.1761	81	10.5849
12	21	1.3222	144	15.8664
$\sum x = 34$		$\sum Y = 5.7536$	$\sum x^2 = 300$	$\sum xY = 40.8862$

Substituting the values of $\sum x$, etc. and calculated by means of above table in the normal equations, we get

$$5.7536 = 5A + 34B$$

and
$$40.8862 = 34A + 300B$$

On solving these equations, we obtain,

$$A = 0.9766\,;\; B = 0.02561$$

Therefore $\quad a = \text{antilog}_{10} A = 9.4754\,;\; b = \dfrac{B}{\log_{10} e} = 0.059$

Hence the required curve is $y = 9.4754e^{0.059x}$. **Ans.**

Example 13: For the data given below, find the equation to the best fitting exponential curve of the form $y = ae^{bx}$.

x	1	2	3	4	5	6
y	1.6	4.5	13.8	40.2	125	300

Sol. $y = ae^{bx}$

On taking log both the sides, $\log y = \log a + bx \log e$ which is of the form $Y = A + Bx$, where $Y = \log y$, $A = \log a$ and $B = b \log e$.

x	y	$Y = \log y$	x^2	xY
1	1.6	0.2041	1	0.2041
2	4.5	0.6532	4	1.3064
3	13.8	1.1399	9	3.4197
4	40.2	1.6042	16	6.4168
5	125	2.0969	25	10.4845
6	300	2.4771	36	14.8626
$\sum x = 21$		$\sum Y = 8.1754$	$\sum x^2 = 91$	$\sum xY = 36.6941$

Normal equations are: $\sum Y = 6A + B\sum x$

$$\sum xY = A\sum x + B\sum x^2$$

Therefore from these equations, we have

$$8.1754 = 6A + 21B$$

$$36.6941 = 21A + 91B$$

$\Rightarrow$ $$A = -0.2534, B = 0.4617$$

Therefore, $a = \text{antilog}A = \text{antilog}(-0.2534) = \text{antilog}(\bar{1}.7466) = 0.5580.$

and $$b = \frac{B}{\log e} = \frac{0.4617}{0.4343} = 1.0631$$

Hence required equation is $y = 0.5580e^{1.0631x}$. **Ans.**

Example 14: Given the following experimental values:

x	0	1	2	3
y	2	4	10	15

Fit by the method of least squares a parabola of the type $y = a + bx^2$.

Sol. Error of estimate for ith point (x_i, y_i) is $e_i = (y_i - a - bx_i^2)$

By the principle of least squares, the values of a and b are such that

$$S = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (y_i - a - bx_i^2)^2 \text{ is minimum.}$$

Therefore normal equations are given by

$$\frac{\partial S}{\partial a} = 0 \Rightarrow \sum y = na + b\sum x^2 \quad ...(1)$$

and $$\frac{\partial S}{\partial b} = 0 \Rightarrow \sum x^2 y = a\sum x^2 + b\sum x^4 \quad ...(2)$$

x	y	x^2	$x^2 y$	x^4
0	2	0	0	0
1	4	1	4	1
2	10	4	40	16
3	15	9	135	81
Total	$\sum y = 31$	$\sum x^2 = 14$	$\sum x^2 y = 179$	$\sum x^4 = 98$

Here $n = 4$.

From (1) and (2), $31 = 4a + 14b$ and $179 = 14a + 98b$

Solving for a and b, we get

$$a = 2.71 \text{ and } b = 1.44$$

Hence the required curve is $y = 2.71 + 1.44x^2$. **Ans.**

Example 15: By the method of least square, find the curve $y = ax + bx^2$ that best fits the following data:

x	1	2	3	4	5
y	1.8	5.1	8.9	14.1	19.8

Sol. Error of estimate for ith point (x_i, y_i) is $e_i = (y_i - ax_i - bx_i^2)$

By the principle of least squares, the values of a and b are such that

$$S = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (y_i - ax_i - bx_i^2)^2 \text{ is minimum.}$$

Therefore normal equations are given by

$$\frac{\partial S}{\partial a} = 0 \Rightarrow \sum_{i=1}^{n} x_i y_i = a\sum_{i=1}^{n} x_i^2 + b\sum_{i=1}^{n} x_i^3 \quad \text{and} \quad \frac{\partial S}{\partial b} = 0 \Rightarrow \sum_{i=1}^{n} x_i^2 y_i = a\sum_{i=1}^{n} x_i^3 + b\sum_{i=1}^{n} x_i^4$$

Dropping the suffix i, normal equations are

$$\sum xy = a\sum x^2 + b\sum x^3 \quad \text{...(1)}$$

and

$$\sum x^2 y = a\sum x^3 + b\sum x^4 \quad \text{...(2)}$$

x	y	x^2	x^3	x^4	xy	x^2y
1	1.8	1	1	1	1.8	1.8
2	5.1	4	8	16	10.2	20.4
3	8.9	9	27	81	26.7	80.1
4	14.1	16	64	256	56.4	225.6
5	19.8	25	125	625	99	495
Total		$\sum x^2 = 55$	$\sum x^3 = 225$	$\sum x^4 = 979$	$\sum xy = 194.1$	$\sum x^2 y = 822.9$

Substituting these values in equations (1) and (2), we get

$$194.1 = 55a + 225b \text{ and } 822.9 = 225a + 979b$$

$$\Rightarrow \quad a = \frac{83.85}{55} \approx 1.52$$

and

$$b = \frac{317.4}{664} \approx 0.49$$

Hence required parabolic curve is $y = 1.52x + 0.49x^2$. **Ans.**

Example 16: Fit an exponential curve of the form $y = ab^x$ to the following data:

x	1	2	3	4	5	6	7	8
y	1.0	1.2	1.8	2.5	3.6	4.7	6.6	9.1

Sol. $y = ab^x$ takes the form $Y = A + Bx$, where $Y = \log y$; $A = \log a$ and $B = \log b$.
Hence the normal equations are given by

$$\sum Y = nA + B\sum x \text{ and } \sum xY = A\sum x + \sum x^2 .$$

x	y	$Y = \log y$	xY	x^2
1	1.0	0.0000	0.000	1
2	1.2	0.0792	0.1584	4
3	1.8	0.2553	0.7659	9
4	2.5	0.3979	1.5916	16
5	3.6	0.5563	2.7815	25
6	4.7	0.6721	4.0326	36
7	6.6	0.8195	5.7365	49
8	9.1	0.9590	7.6720	64
$\sum x = 36$	$\sum y = 30.5$	$\sum Y = 3.7393$	$\sum xY = 22.7385$	$\sum x^2 = 204$

Putting the values in the normal equations, we obtain

$3.7393 = 8A + 36B$ and $22.7385 = 36A + 204B$

$\Rightarrow$ $B = 0.1407$ and $A = 0.1656$

$\Rightarrow$ $b = \text{antilog}B = 1.38$ and $a = \text{antilog}\, A = 0.68.$

Thus the required curve of best fit is $y = (0.68)(1.38)^x$. **Ans.**

Example 17: Fit a curve $y = ab^x$ to the following data:

x	2	3	4	5	6
y	144	172.8	207.4	248.8	298.5

Sol. Given equation $y = ab^x$ reduces to $Y = A + Bx$ where $Y = \log y$, $A = \log a$ and $B = \log b$.
The normal equations are,

$$\sum \log y = n \log a + \log b \sum x$$

$$\sum x \log y = \log a \sum x + \log b \sum x^2$$

The calculations of $\sum x$, $\sum \log y$, $\sum x^2$ and $\sum x \log y$ are substitute in the following tabular form.

x	y	x^2	$\log y$	$x \log y$
2	144	4	2.1584	4.3168
3	172.8	9	2.2375	6.7125
4	207.4	16	2.3168	9.2672
5	248.8	25	2.3959	11.9795
6	298.5	36	2.4749	14.8494
20		90	11.5835	47.1254

Putting these values in the normal equations, we have

$$11.5835 = 5 \log a + 20 \log b$$

$$47.1254 = 20 \log a + 90 \log b.$$

Solving these equations and taking antilog, we have a = 100, b = 1.2 approximate. Therefore equation of the curve is $y = 100(1.2)^x$. **Ans.**

Example 18: Derive the least square equations for fitting a curve of the type $y = ax^2 + (b/x)$ to a set of n points. Hence fit a curve of this type to the data.

x	1	2	3	4
y	−1.51	0.99	3.88	7.66

Sol. Let the n points are given by (x_1, y_1), (x_2, y_2), (x_3, y_3), ……, (x_n, y_n). The error of estimate for the ith point (x_i, y_i) is $e_i = [y_i - ax_i^2 - (b/x_i)]$.

By the principle of least square, the values of a and b are such so that the sum of the square of error S, *viz.*,

$$S = \sum_{i=1}^{n} e_i^2 = \sum \left(y_i - ax_i^2 - \frac{b}{x_i} \right)^2 \text{ is minimum.}$$

Therefore the normal equations are given by

$$\frac{\partial S}{\partial a} = 0, \frac{\partial S}{\partial b} = 0$$

or

$$\sum_{i=1}^{n} y_i x_i^2 = a \sum_{i=1}^{n} x_i^4 + b \sum_{i=1}^{n} x_i \text{ and } \sum_{i=1}^{n} \frac{y_i}{x_i} = a \sum_{i=1}^{n} x_i + b \sum_{i=1}^{n} \frac{1}{x_i^2}$$

These are the required least square equations.

x	y	x^2	x^4	$\frac{1}{x}$	$\frac{1}{x^2}$	yx^2	$\frac{y}{x}$
1	−1.51	1	1	1	1	−1.51	−1.51
2	0.99	4	16	0.5	0.25	3.96	0.495
3	3.88	9	81	0.3333	0.1111	34.92	1.2933
4	7.66	16	256	0.25	0.0625	122.56	1.0943
10			354		1.4236	159.93	1.1933

Putting the values in the above least square equations, we get

$$159.93 = 354a + 10b \text{ and } 1.1933 = 10a + 1.4236b.$$

Solving these, we get a = 0.509 and b = –2.04.

Therefore, the equation of the curve fitted to the above data is $y = 0.509x^2 - \frac{2.04}{x}$. **Ans.**

Example 19: Fit the curve $pv^\gamma = k$ to the following data:

$p(\text{kg/cm}^2)$	0.5	1	1.5	2	2.5	3
v (litres)	1620	1000	750	620	520	460

Sol. Given $pv^\gamma = k$

$$v = \left(\frac{k}{p}\right)^{1/\gamma} = k^{1/\gamma} p^{-1/\gamma}$$

Taking log both the sides, we get

$$\log v = \frac{1}{\gamma}\log k - \frac{1}{\gamma}\log p$$

which is of the form $Y = A + BX$

where $Y = \log v$, $X = \log p$, $A = \frac{1}{\gamma}\log k$ and $B = -\frac{1}{\gamma}$.

p	v	X	Y	XY	X^2
0.5	1620	–0.30103	3.20952	–0.96616	0.09062
1	1000	0	3	0	0
1.5	750	0.17609	2.87506	0.50627	0.03101
2	620	0.30103	2.79239	0.84059	0.09062
2.5	520	0.39794	2.716	1.08080	0.15836
3	460	0.47712	2.66276	1.27046	0.22764
Total		$\sum X = 1.05115$	$\sum Y = 17.25573$	$\sum XY = 2.73196$	$\sum X^2 = 0.59825$

Here n = 6

Normal equations are,

$$17.25573 = 6A + 1.05115B$$
$$2.73196 = 1.05115A + 0.59825B$$

On solving these, we get

$$A = 2.99911 \text{ and } B = -0.70298$$

$\therefore \qquad \gamma = -\frac{1}{B} = \frac{1}{0.70298} = 1.42252$

Again $\qquad \log k = \gamma A = 4.26629$

$\therefore \qquad k = \text{antilog}(4.26629) = 18462.48$

Hence, the required curve is $pv^{1.42252} = 18462.48.$ **Ans.**

Example 20: For the data given below, find the equation to the best fitting exponential curve of the form $y = ae^{bx}$.

x	1	2	3	4	5	6
y	1.6	4.5	13.8	40.2	125	300

Sol. Given $y = ae^{bx}$, taking log we get $\log y = \log a + bx \log_{10} e$ which is of the $Y = A + Bx$,

where $\qquad Y = \log y,\ A = \log a$ and $B = \log_{10} e$.

Put the values in the following tabular form, also transfer the origin of x series to 3, so that $u = x - 3$.

x	y	$\log y = Y$	$x - 3 = u$	uY	u^2
1	1.6	0.204	−2	−0.408	4
2	4.5	0.653	−1	−0.653	1
3	13.8	1.140	0	0	0
4	40.2	1.604	1	1.604	1
5	125.0	2.094	2	4.194	4
6	300	2.477	3	7.431	9
Total		8.175	3	12.168	19

In case $Y = A + B.u$, then normal equations are given by

$$\sum Y = nA + B\sum u \quad \Rightarrow \quad 8.175 = 6A + 3B \qquad \ldots(1)$$

$$\sum uY = A\sum u + B\sum u^2 \quad \Rightarrow \quad 12.168 = 3A + 19B \qquad \ldots(2)$$

Solving (1) and (2), we get

$$A = 1.13 \text{ and } B = 0.46$$

Thus equation is $Y = 1.13 + 0.46u$, *i.e.* $Y = 1.13 + 0.46(x - 3)$

or $\qquad Y = 0.46x - 0.25$

Which gives $\log a = -0.25$ *i.e.* $\text{antilog}(-0.25) = \text{antilog}(\bar{1}.75) = 0.557$

$$b = \frac{B}{\log_{10} e} = \frac{.46}{0.4343} = 1.06$$

Hence, the required equation of the curve is $y = (0.557)e^{1.06x}$. **Ans.**

PROBLEM SET 5.1

1. Fit a straight line to the given data regarding x as the independent variable:

x	1	2	3	4	6	8
y	2.4	3.1	3.5	4.2	5.0	6.0

[**Ans.** $y = 2.0253 + 0.502x$]

2. Fit a straight line $y = a + bx$ to the following data by the method of least square:

x	0	1	3	6	8
y	1	3	2	5	4

[**Ans.** $1.6 + 0.38x$]

3. Find the least square approximation of the form $y = a + bx^2$ for the following data:

x	0	0.1	0.2	0.3	0.4	0.5
y	1	1.01	0.99	0.85	0.81	0.75

[**Ans.** $y = 1.0032 - 1.1081x^2$]

4. Fit a second degree parabola to the following data:

x	0.0	1.0	2.0
y	1.0	6.0	17.0

[**Ans.** $y = 1 + 2x + 3x^2$]

5. Fit a second degree parabola to the following data:

x	1.0	1.5	2.0	2.5	3.0	3.5	4.0
y	1.1	1.3	1.6	2.0	2.7	3.4	4.1

[**Ans.** $y = 1.04 - 0.193x + 0.243x^2$]

6. Fit a second degree parabola to the following data by the least square method:

x	1	2	3	4	5
y	1090	1220	1390	1625	1915

[**Ans.** $y = 27.5x^2 + 40.5x + 1024$]

7. Fit a parabola $y = a + bx + cx^2$ to the following data:

x	2	4	6	8	10
y	3.07	12.85	31.47	57.38	91.29

[**Ans.** $y = 0.34 - 0.78x + 0.99x^2$]

8. Determine the constants a and b by the method of least squares such that $y = ae^{bx}$ fits the following data:

x	2	4	6	8	10
y	4.077	11.084	30.128	81.897	222.62

[**Ans.** $y = 1.49989e^{0.50001x}$]

9. Fit a least square geometric curve $y = ax^b$ to the following data:

x	1	2	3	4	5
y	0.5	2	4.5	8	12.5

[**Ans.** $y = 0.5012x^{1.9977}$]

10. A person runs the same race track for five consecutive days and is timed as follows:

Day(x)	1	2	3	4	5
Time(y)	15.3	15.1	15	14.5	14

Make a least square fit to the above data using a function $a+\frac{b}{x}+\frac{c}{x^2}$.

$\left[\textbf{Ans. } y = 13.0065 + \frac{6.7512}{x} + \frac{4.4738}{x^2}\right]$

11. Use the method of least squares to fit the curve $y=\frac{c_0}{x}+c_1\sqrt{x}$ to the following table of values:

x	0.1	0.2	0.4	0.5	1	2
y	21	11	7	6	5	6

$\left[\textbf{Ans. } y = \frac{1.97327}{x} + 3.28182\sqrt{x}\right]$

12. Using the method of least square to fit a parabola $y = a+bx+cx^2$ in the following data:

$(x, y): (-1, 2), (0, 0), (0, 1), (1, 2)$

$\left[\textbf{Ans. } y = \frac{1}{2} + \frac{3}{2}x^2\right]$

13. The pressure of the gas corresponding to various volumes V is measured, given by the following data:

$V(\text{cm}^3)$	50	60	70	90	100
$p(\text{kgcm}^{-2})$	64.7	51.3	40.5	25.9	78

Fit the data to the equation $pV^{\gamma}=c$. [**Ans.** $pV^{0.28997} = 167.78765$]

14. Employ the method of least squares to fit a parabola $y = a + bx + cx^2$ in the following data: (x, y): (–1, 2), (0, 0), (0, 1), (1, 2) [**Ans.** $y = 0.5 + 1.5x^2$]

15. Fit a second degree parabola in the following data: [*U.T.U. 2008*]

x	0.0	1.0	2.0	3.0	4.0
y	1.0	4.0	10.0	17.0	30.0

[**Ans.** $y = 1 + 2x + 3x^2$]

16. Fit at least square quadratic curve to the following data:

x	1	2	3	4
y	1.7	1.8	2.3	3.2

, estimate $y(2.4)$

[**Ans.** $y = 2 - 0.5x + 0.2x^2$ and $y(2.4) = 1.952$]

17. Fit an exponential curve by least squares

x	1	2	5	10	20	30	40	50
y	98.2	91.7	81.3	64.0	36.4	32.6	17.1	11.3

Estimate y when $x = 25$. **[Ans.** $y = 100(0.96)^x$, $y(25) = 33.9$**]**

18. Fit the curve $y = a + \frac{b}{x}$ to the following data

x	1	2	3	4
y	3	1.5	6	7.5

Estimate y when $x = 2.25$. $\left[\textbf{Ans. } y = 1.3 + \frac{1.7}{x}, \; y(2.25) = 4.625\right]$

5.10 POLYNOMIAL

If $$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots\ldots\ldots\ldots + a_{n-1}x + a_n$$

Then the above relation is called the polynominal of nth order in x.

5.10.1 Degree of Polynomial

The highest power of x occurring in the given polynomial is called degree of polynomial.

The constant $c = cx^0$ is called a polynomial of degree zero.

The polynomial $f(x) = ax + b, a \neq 0$ is of degree one and is called a linear polynomial.

The polynomial $f(x) = ax^2 + bx + c, \; a \neq 0$ is of degree two and is called a quadratic polynomial.

The polynomial $f(x) = ax^3 + bx^2 + cx + d, \; a \neq 0$ is of degree three and is called a cubic polynomial.

The polynomial $f(x) = ax^4 + bx^3 + cx^2 + dx + e, \; a \neq 0$ is of degree four and is called biquadratic polynomial.

5.11 DESCARTE'S RULE OF SIGNS

The number of positive roots of the equation $f(x) = 0$ cannot exceed the number of changes of sign in $f(x)$, and the number of negative roots cannot exceed the number of changes of sign of $f(-x)$.

Existence of imaginary roots: If an equation of the nth degree has at most p positive roots and at most q negative roots, then it has at least $n - (p + q)$ imaginary roots.

Example 1: Apply Descarte's Rule of signs to discuss the nature of the roots of the equation $x^4 + 15x^2 + 7x - 11 = 0$.

Sol. The given equation is $f(x) = x^4 + 15x^2 + 7x - 11 = 0$

Signs of $f(x)$ are + + + – [from + to –]

It has one change of sign and hence it must have one +ve root.

$$f(-x) = (-x)^4 + 15(-x)^2 + 7(-x) - 11 = 0$$

or $$f(-x) = x^4 + 15x^2 - 7x - 11 \qquad = 0$$

Signs of $f(-x)$ are + + – – [from + to –]

It has only one change in sign and hence it must have one –ve root.

Thus the equation has two real roots , one +ve and one –ve and hence the other two roots must be imaginary.

Example 2: Show that $x^7 - 3x^4 + 2x^3 - 1 = 0$ has at least four imaginary roots.

Sol. The given equation is

$$f(x) = x^7 - 3x^4 + 2x^3 - 1 = 0 \qquad \text{[from + to – or – to +]}$$

Signs of $f(x)$ + – + –

$\therefore f(x) = 0$ has 3 changes of sign. Therefore, it cannot have more than three positive roots.

Also $$f(-x) = (-x)^7 - 3(-x)^4 + 2(-x)^3 - 1 = 0$$

or $$f(-x) = -x^7 - 3x^4 - 2x^3 - 1 = 0$$

Signs of $f(x)$ are – – – –

$\therefore f(-x) = 0$ has no changes in sign. Therefore the given equation has no negative root.

Thus the given equation cannot have more than 3 + 0 = 3 real roots. But the given equation has 7 roots. Hence the given equation has 7 – 3 = 4 imaginary roots.

Example 3: Find the least positive number of imaginary roots of the equation

$$f(x) = x^9 - x^5 + x^4 + x^2 + 1 = 0$$

Sol. The given equation is $f(x) = x^9 - x^5 + x^4 + x^2 + 1 = 0$

Signs of $f(x)$ are + – + + +

$\therefore$ $f(x) = 0$ has two changes of signs, and hence 2 is the max. number of +ve root.

$$f(-x) = (-x)^9 - (-x)^5 + (-x)^4 + (-x)^2 + 1 = 0$$

or $$f(-x) = -x^9 + x^5 + x^4 + x^2 + 1 = 0$$

Signs of $f(-x)$ are – + + + +

$\therefore f(-x) = 0$ has only one changes of sign and hence it has only one –ve root or $f(x) = 0$ has only one –ve root.

Thus the max. number of real roots is 2 + 1=3 and the equation being of 9th degree and it will have at least 9 – 3 = 6 imaginary roots.

5.12 CARDON'S METHOD

Case 1. When Cubic is of the Form $x^3 + qx + r = 0$

The given cubic is $x^3 + qx + r = 0$ ...(1)

Let $x = u + v$ be a root of (1)

Cubing, $x^3 = u^3 + v^3 + 3uv(u+v) = u^3 + v^3 + 3uvx$

or $x^3 - 3uvx - (u^3 + v^3) = 0$ ...(2)

Comparing (1) and (2), $uv = -\frac{1}{3}q$ or $u^3v^3 = -\frac{1}{27}q^3$ and $u^3 + v^3 = -r$

$\therefore$ u^3 and v^3 are the roots of the equation

$$t^2 - (u^3 + v^3)\, t + u^3v^3 = 0 \quad \text{or} \quad t^2 + rt - \frac{1}{27}q^3 = 0 \qquad ...(3)$$

Solving (3),

$$t = \frac{-r \pm \sqrt{r^2 + \frac{4q^3}{27}}}{2}$$

Let

$$u^3 = \frac{-r + \sqrt{r^2 + \frac{4q^3}{27}}}{2}; v^3 = \frac{-r - \sqrt{r^2 + \frac{4q^3}{27}}}{2}$$

Now, the three cube roots of u^3 are $u, u\omega, u\omega^2$ and those of v^3 are $v, v\omega, v\omega^2$, where ω and ω^2 are imaginary cube root of unity.

Since $x = u + v$

To find x, we have to add a cube root of u^3 and a cube of v^3 in such a manner that their product is real.

$\therefore$ The three values of x are $u + v,\ u\omega + v\omega^2,\ u\omega^2 + v\omega$ $(\because \omega^3 = 1)$

Example 4: Use Cardon's method to solve

$$x^3 - 27x + 54 = 0.$$

Sol. Let $x = u + v$

Cubing, $x^3 = (u+v)^3 = u^3 + v^3 + 3uv(u+v) = u^3 + v^3 + 3uvx$

$\Rightarrow$ $x^3 - 3uvx - (u^3 + v^3) = 0$

Comparing with the given equation,we get

$$uv = 9 \Rightarrow u^3v^3 = 729 \quad \text{(on cubing)}$$

and $u^3 + v^3 = -54$

$\therefore$ u^3 and v^3 are the root of $t^2 - (u^3 + v^3)t + u^3v^3 = 0$

$\Rightarrow$ $t^2 + 54t + 729 = 0 \;\Rightarrow\; (t + 27)^2 = 0$

$\Rightarrow$ $t = -27, \; -27$

Let $u^3 = -27$ and $v^3 = -27$

So that, $u = -3, -3\omega, -3\omega^2$ and $v = -3, -3\omega, -3\omega^2$

To find x, we have to add a cube root of u^3 and a cube root of v^3 in such a way that their product is real.

$$\therefore \quad x = (-3 - 3), (-3\omega - 3\omega^2), (-3\omega^2 - 3\omega) \qquad (\because \omega^3 = 1)$$

$$= -6, \; -3(\omega + \omega^2), \; -3(\omega^2 + \omega) = -6, 3, 3 \qquad (\because 1 + \omega + \omega^2 = 0)$$

Hence the required roots are –6, 3, 3. **Ans.**

Example 5: Solve by Cardon's method $x^3 - 15x - 126 = 0$.

Sol. Let $x = u + v$

Cubing, $x^3 = (u + v)^3 = u^3 + v^3 + 3uv(u + v) = u^3 + v^3 + 3uvx$

$\Rightarrow$ $x^3 - 3uvx - (u^3 + v^3) = 0$

Comparing with the given equation, we get

$$uv = 5 \Rightarrow u^3v^3 = 125 \text{ (on cubing)}$$

and $u^3 + v^3 = 126$

$\therefore$ u^3 and v^3 are the roots of $t^2 - (u^3 + v^3)t + u^3v^3 = 0$

$\Rightarrow$ $t^2 - 126t + 125 = 0 \;\Rightarrow\; (t - 1)(t - 125) = 0 \;\Rightarrow\; t = 1, 125$

Let $u^3 = 1$ and $v^3 = 125$

So that, $u = 1, \; \omega, \omega^2$ and $v = 5, \; 5\omega, 5\omega^2$

To find x, we have to add a cube root of u^3 and a cube root of v^3 in such a way that their product is real.

$$\therefore \quad x = 1 + 5, \omega + 5\omega^2, \omega^2 + 5\omega \qquad (\because \omega^3 = 1)$$

$$= 6, \left\{\left(\frac{-1 + i\sqrt{3}}{2}\right) + 5\left(\frac{-1 - i\sqrt{3}}{2}\right)\right\}, \left\{\left(\frac{-1 - i\sqrt{3}}{2}\right) + 5\left(\frac{-1 + i\sqrt{3}}{2}\right)\right\}$$

$$= 6, -3 - 2\sqrt{3}i, -3 + 2\sqrt{3}i = 6, -3 \pm 2\sqrt{3}i$$

Hence, the required roots are $6, -3 \pm 2\sqrt{3}i$. **Ans.**

Example 6: Solve $x^3 - 6x - 9 = 0$ by Cardon's method.

Sol. Let $x = u + v$

Cubing, $$x^3 = (u+v)^3 = u^3 + v^3 + 3uv(u+v)$$

$$= u^3 + v^3 + 3uvx$$

$\Rightarrow$ $x^3 - 3uvx - (u^3 + v^3) = 0$

Comparing, we get

$$uv = 2 \Rightarrow u^3v^3 = 8 \text{ (on cubing)}$$

and $u^3 + v^3 = 9$

$\therefore$ u^3, v^3 are the roots of $t^2 - (u^3+v^3)t + u^3v^3 = 0$

$\Rightarrow$ $t^2 - 9t + 8 = 0 \Rightarrow (t-1)(t-8) = 0 \Rightarrow t = 1, 8$

Let $u^3 = 1 \text{ and } v^3 = 8$ so that

$$u = 1, \omega, \omega^2 \text{ and } v = 2, 2\omega, 2\omega^2$$

To find x, we have to add a cube root of u^3 and a cube root of v^3 in such that their product is real.

$\therefore$ $$x = 1 + 2, \omega + 2\omega^2, \omega^2 + 2\omega \qquad (\because \omega^3 = 1)$$

$$= 3, \left\{\left(\frac{-1+i\sqrt{3}}{2}\right) + 2\left(\frac{-1-i\sqrt{3}}{2}\right)\right\}, \left\{\left(\frac{-1-i\sqrt{3}}{2}\right) + 2\left(\frac{-1+i\sqrt{3}}{2}\right)\right\}$$

$$= 3, \frac{-3-i\sqrt{3}}{2}, \frac{-3+i\sqrt{3}}{2} \Rightarrow x = 3, \frac{-3 \pm i\sqrt{3}}{2}.$$ **Ans.**

Example 7: Solve the cubic equation $x^3 - 18x - 35 = 0$

Sol. Let $x = u + v$

Cubing, $$x^3 = (u+v)^3 = u^3 + v^3 + 3uv(u+v)$$

$$= u^3 + v^3 + 3uvx$$

$\Rightarrow$ $x^3 - 3uvx - (u^3 + v^3) = 0$

Comparing, we get $uv = 6 \Rightarrow u^3v^3 = 216$ (on cubing)

and $u^3 + v^3 = 35$

$\therefore$ u^3 and v^3 are the roots of $t^2 - (u^3+v^3)t + u^3v^3 = 0$

$\Rightarrow$ $t^2 - 35t + 216 = 0 \Rightarrow (t-8)(t-27) = 0 \Rightarrow t = 8, 27$

Let $u^3 = 8 \;\&\; v^3 = 27$ so that

$$u = 2, 2\omega, 2\omega^2 \text{ and } v = 3, 3\omega, 3\omega^2$$

To find x, we have to add a cube root of u^3 and a cube root of v^3 in such that their product is real.

$\therefore$ $$x = 2+3,\ 2\omega+3\omega^2,\ 2\omega^2+3\omega \qquad (\because \omega^3 = 1)$$

$$= 5, \left\{2\left(\frac{-1+i\sqrt{3}}{2}\right)+3\left(\frac{-1-i\sqrt{3}}{2}\right)\right\}, \left\{2\left(\frac{-1-i\sqrt{3}}{2}\right)+3\left(\frac{-1+i\sqrt{3}}{2}\right)\right\}$$

$$= 5, \frac{-5-i\sqrt{3}}{2}, \frac{-5+i\sqrt{3}}{2}$$

$\Rightarrow$ $$x = 5, \frac{-5 \pm i\sqrt{3}}{2}.$$ **Ans.**

Case 2. When the Cubic Equation is of the Form $a_0x^3 + a_1x^2 + a_2x + a_3 = 0$

Then, first of all we remove the term containing x^2.

This is done by diminishing the roots of the given equation by $\frac{-a_1}{na_0}$ or Sum of root/No. of roots. Where n is 3. We proceed with the help of following examples:

Example 8: Solve by Cardon's method $x^3 + 6x^2 + 9x + 4 = 0$

Sol. Equationing $x^3 + 6x^2 + 9x + 4 = 0$ ...(1)

Equating with $a_0x^3 + a_1x^2 + a_2x + a_3 = 0,$ we get ...(2)

$$a_0 = 1, a_1 = 6$$

Then $$h = -\frac{a_1}{3a_0} = -\frac{6}{3} = -2$$

$$h = -2.$$

Now remove the x^2 terms ,we have

–2	1	6	9	4
		–2	–8	–2
	1	4	1	(2)
		–2	–4	
	1	2	(–3)	
		–2		
	1	(0)		

The transformed equation is $y^3 - 3y + 2 = 0$ ...(3)

where $$y = x + 2.$$

Let $y = u + v$ be a solution of (3), then we get

$$y^3 - 3uvy - (u^3 + v^3) = 0 \quad ...(4)$$

Equating (3) and (4), we get $uv = 1$

$\Rightarrow$ $$u^3v^3 = 1 \text{ and } u^3 + v^3 = -2$$

Let us consider an equation whose roots are u^3 and v^3

$$t^2 - st + p = 0 \quad s = \text{sum of roots}, p = \text{product of roots}$$

$$t^2 + 2t + 1 = 0 \;\Rightarrow\; (t+1)^2 = 0$$

$\Rightarrow$ $t = -1, -1$

Let $u^3 = -1$ and $v^3 = -1$ so that

$$u = -1, -\omega, -\omega^2 \text{ and } v = -1, -\omega, -\omega^2$$

$\therefore$ $y = u + v = (-1-1), (-\omega - \omega^2), (-\omega^2 - \omega)$

$= -2, 1, 1$ $\qquad (\because 1 + \omega + \omega^2 = 0)$

But $x = y - 2$

$= -2-2, 1-2, 1-2 = -4, -1, -1$

$\therefore$ $x = -4, -1, -1$. **Ans.**

Example 9: Solve by Cardon's method $x^3 - 15x^2 - 33x + 847 = 0.$

Sol. The given equation $x^3 - 15x^2 - 33x + 847 = 0$ ...(1)

Equating with $a_0x^3 + a_1x^2 + a_2x + a_3 = 0,$ we get

$$a_0 = 1, a_1 = -15$$

$\therefore$ $h = -\dfrac{a_1}{3a_0} = \dfrac{15}{3} = 5$

Now remove the x^2 term, using synthetic division, we have

5	1	–15	–33	847
		5	–50	–415
	1	–10	–83	(432)
		5	–25	
	1	–5	(–108)	
		5		
	1	(0)		

Transformed equation is

$$y^3 - 108y + 432 = 0 \qquad ...(2)$$

where $y = x - 5.$

Let $y = u + v$

$\therefore$ $y^3 - 3uvy - (u^3 + v^3) = 0$ ...(3) (on cubing)

Comparing (2) and (3),

$$uv = 36 \;\Rightarrow\; u^3v^3 = (6)^6 \text{ and } u^3 + v^3 = -432$$

$\therefore$ u^3 and v^3 are the roots of $t^2 + 432t + (6)^6 = 0$

$\Rightarrow$ $t^2 + 2.(6)^3 t + (6)^6 = 0 \;\Rightarrow\; (t+216)^2 = 0 \;\Rightarrow\; t = -216, \; -216$

Let $u^3 = -216$ and $v^3 = -216$ so that

$$u = -6, -6\omega, -6\omega^2 \text{ and } v = -6, -6\omega, -6\omega^2$$

$\therefore$ $y = u + v = (-6-6), (-6\omega - 6\omega^2), (-6\omega^2 - 6\omega)$

$= -12, 6, 6$ $\left(\begin{array}{r} \because \omega + \omega^2 + 1 = 0 \\ \omega + \omega^2 = -1 \end{array}\right)$

But $x = y + 5 = -12 + 5, 6 + 5, 6 + 5$

$\therefore$ $x = -7, 11, 11.$ **Ans.**

Example 10: Solve $x^3 - 3x^2 + 12x + 16 = 0$

Sol. Here $a_0 = 1, a_1 = -3$ $\therefore$ $h = -\frac{a_1}{3a_0} = \frac{3}{3} = 1$

Now remove the term x^2, using synthetic division

1	1	–3	12	16
		1	–2	10
	1	–2	10	(26)
		1	–1	
	1	–1	(9)	
		1		
	1	(0)		

The transformed equation is

$$(x-1)^3 + 9(x-1) + 26 = 0$$

Let $y = x - 1$, then $y^3 + 9y + 26 = 0$ ...(1)

Let $y = u + v$ ...(2)

$\Rightarrow$ $y^3 - 3uv(y) - (u^3 + v^3) = 0$

Comparing (1) and (2), we get

$$uv = -3 \text{ and } u^3 + v^3 = -26$$

$\Rightarrow$ $u^3v^3 = -27$

Now let us have an equation whose roots are u^3 and v^3

$$t^2 - (-26)t + (-27) = 0 \;\Rightarrow\; t^2 + 26t - 27 = 0$$

$$\Rightarrow \qquad (t+27)(t-1) = 0$$

$$\Rightarrow \qquad t = 1, -27 \quad i.e., \quad u^3 = 1 \quad \text{and} \quad v^3 = -27$$

So that $\qquad u = 1, \omega, \omega^2 \quad \text{and} \quad v = -3, -3\omega, -3\omega^3$

$$y = u + v = (1-3), (\omega - 3\omega^2), (\omega^2 - 3\omega)$$

$$= -2, \left\{\frac{-1+i\sqrt{3}}{2} - 3\left(\frac{-1-i\sqrt{3}}{2}\right)\right\}, \left\{\frac{-1-i\sqrt{3}}{2} - 3\left(\frac{-1+i\sqrt{3}}{2}\right)\right\}$$

$$= -2, \left\{(1+2i\sqrt{3})\right\}, \left\{(1-2i\sqrt{3})\right\}$$

$$\therefore \qquad x = y + 1$$

$$\Rightarrow \qquad x = (-2+1), (1+2i\sqrt{3}+1), (1-2i\sqrt{3}+1)$$

$$= -1, (2+2i\sqrt{3}), (2-2i\sqrt{3})$$

$$= -1, 2(1 \pm i\sqrt{3}). \qquad \textbf{Ans.}$$

Example 11: Solve $x^3 - 6x^2 + 6x - 5 = 0$ by Cardon's method.

Sol. The given equation $\quad x^3 - 6x^2 + 6x - 5 = 0 \qquad$ ...(1)

Comparing with $\; a_0x^3 + a_1x^2 + a_2x + a_3 = 0,$ we get

$$a_0 = 1, a_1 = -6$$

$$\therefore \qquad h = -\frac{a_1}{3a_0} = \frac{6}{3} = 2$$

Now remove the x^2 term, using synthetic division

2	1	−6	6	−5
		2	−8	−4
	1	−4	−2	(−9)
		2	−4	
	1	−2	(−6)	
		2		
	1	(0)		

Transformed equation is $\; y^3 - 6y - 9 = 0 \qquad$ ...(2)

where $$y = x - 2$$

Let $y = u + v$ be the solution of (2), then

$$y^3 = u^3 + v^3 + 3uv\,(u+v)$$

$$y^3 = u^3 + v^3 + 3uv\,(y)$$

$$y^3 - 3uv(y) - (u^3 + v^3) = 0 \qquad \ldots(3)$$

Comparing (2) and (3), we have

$$3uv = 6 \;\Rightarrow\; uv = 2 \;\Rightarrow\; u^3v^3 = 8$$

and $$u^3 + v^3 = 9$$

Now let us have an equation whose roots are u^3 and v^3

$t^2 - st + p = 0$; s = sum of roots, p = product of the roots.

$$t^2 - 9t + 8 = 0$$

$t = 1, 8$ *i.e.*, $u^3 = 1$ and $v^3 = 8$ so that

$$u = 1, \omega, \omega^2 \text{ and } v = 2, 2\omega, 2\omega^2$$

$\therefore$ $$y = u + v$$

$$= (1+2), (\omega + 2\omega^2), (\omega^2 + 2\omega)$$

$$= 3, \left\{\left(\frac{-1+i\sqrt{3}}{2}\right) + 2\left(\frac{-1-i\sqrt{3}}{2}\right)\right\}, \left\{\left(\frac{-1-i\sqrt{3}}{2}\right) + 2\left(\frac{-1+i\sqrt{3}}{2}\right)\right\}$$

$$= 3, \frac{-3-i\sqrt{3}}{2}, \frac{-3+i\sqrt{3}}{2}.$$

But $$x = y + 2$$

$$= 5, \frac{1-i\sqrt{3}}{2}, \frac{1+i\sqrt{3}}{2}$$

$$= 5, \frac{1}{2}(1 \pm i\sqrt{3}).$$ **Ans.**

Example 12: Solve the equation $x^3 - 3x^2 + 3x - 1 = 0$ by Cardon's method.

Sol. Here, $a_0 = 1, a_1 = -3$

$\therefore$ $$h = -\frac{a_1}{3a_0} = \frac{3}{3} = 1$$

Now remove the x^2 term, using synthetic division, we have

1	1	−3	3	−1
		1	−2	1
	1	−2	1	(0)
		1	−1	
	1	−1	(0)	
		1		
	1	(0)		

Transformed equation is $y^3 = 0$ ...(1)

where $y = x - 1.$

From (1), $y = 0, 0, 0$

$\therefore$ $x = y + 1 = 1, 1, 1$

Hence the required roots are 1, 1, 1. **Ans.**

Example 13: Solve the cubic equation $x^3 + x^2 - 16x + 20 = 0.$

Sol. Given equation is $x^3 + x^2 - 16x + 20 = 0$

Here, $a_0 = 1, a_1 = 1$

$\therefore$ $h = -\dfrac{a_1}{3a_0} = -\dfrac{1}{3}$

Now remove the x^2 term, using synthetic division, we have

$-1/3$	1	1	−16	20
		$-1/3$	$-2/9$	$\frac{146}{27}$
	1	$2/3$	$\frac{-146}{9}$	$\left(\frac{686}{27}\right)$
		$-1/3$	$-1/9$	
	1	$1/3$	$\left(\frac{-147}{9}\right)$	
		$-1/3$		
	1	(0)		

Transformed equation is $y^3 - \dfrac{147}{9}y + \dfrac{686}{27} = 0$ ...(1)

where $y = x + \dfrac{1}{3}.$

Let $y = u + v$

Cubing, $y^3 = u^3 + v^3 + 3uvy$

$\Rightarrow \quad y^3 - 3uvy - (u^3 + v^3) = 0$ ...(2)

Comparing with (1), we get

$$uv = \frac{49}{9} \Rightarrow u^3v^3 = \left(\frac{7}{3}\right)^6 \text{ and } u^3 + v^3 = -\frac{686}{27}$$

$\therefore$ u^3 and v^3 are the roots of $t^2 + \frac{686}{27}t + \left(\frac{7}{3}\right)^6 = 0$

$$t^2 + 2\left(\frac{7}{3}\right)^3 t + \left(\frac{7}{3}\right)^6 = 0$$

$\Rightarrow$ $$\left\{t + \left(\frac{7}{3}\right)^3\right\}^2 = 0 \Rightarrow t = -\left(\frac{7}{3}\right)^3, -\left(\frac{7}{3}\right)^3$$

Let $$u^3 = -\left(\frac{7}{3}\right)^3 \text{ and } v^3 = -\left(\frac{7}{3}\right)^3$$

So that $$u = \frac{-7}{3}, -\frac{7}{3}\omega, -\frac{7}{3}\omega^2 \text{ and } v = -\frac{7}{3}, -\frac{7}{3}\omega, -\frac{7}{3}\omega^2$$

$\therefore$ $$y = u + v = \left(-\frac{7}{3} - \frac{7}{3}\right), \left(-\frac{7}{3}\omega - \frac{7}{3}\omega^2\right), \left(-\frac{7}{3}\omega^2 - \frac{7}{3}\omega\right) = -\frac{14}{3}, \frac{7}{3}, \frac{7}{3}$$

Now, $$x = y - \frac{1}{3} = \frac{-14}{3} - \frac{1}{3}, \frac{7}{3} - \frac{1}{3}, \frac{7}{3} - \frac{1}{3} = -5, 2, 2$$

Hence required roots are –5, 2, 2. **Ans.**

Example 14: Solve the cubic equation $x^3 + 6x^2 - 12x + 32 = 0.$

Sol. Given equation is $x^3 + 6x^2 - 12x + 32 = 0$

Here, $a_0 = 1, a_1 = 6$

$\therefore$ $$h = -\frac{a_1}{3a_0} = -\frac{6}{3} = -2$$

Now remove the x^2 term, using synthetic division method, we have

–2	1	6	– 12	32
		–2	– 8	40
	1	4	– 20	(72)
		–2	– 4	
	1	2	(–24)	
		–2		
	1	(0)		

Transformed equation is

$$y^3 - 24y + 72 = 0 \quad \text{...(1)}$$

where $y = x + 2$

Let $y = u + v$

Cubing, $y^3 = u^3 + v^3 + 3uvy$

$\Rightarrow$ $y^3 - 3uvy - (u^3 + v^3) = 0$ ...(2)

Comparing (1) and (2), we get

$$uv = 8 \;\Rightarrow\; u^3v^3 = 512 \text{ and } u^3 + v^3 = -72$$

$\therefore$ u^3 and v^3 are the roots of $t^2 + 72t + 512 = 0$

$\Rightarrow$ $t = -8, -64$

Let $u^3 = -8$ and $v^3 = -64$

So that $u = -2, -2\omega, -2\omega^2$ & $v = -4, -4\omega, -4\omega^2$

$\therefore$ $y = u + v = (-2-4), (-2\omega - 4\omega^2), (-2\omega^2 - 4\omega)$

$= -6, 3 + i\sqrt{3}, 3 - i\sqrt{3}$

Now, $x = y - 2 = -8, 1 + i\sqrt{3}, 1 - i\sqrt{3}$

Hence the required roots are $-8, 1 \pm i\sqrt{3}$. **Ans.**

Example 15: Solve $x^3 + 3ax^2 + 3(a^2 - bc)x + a^3 + b^3 + c^3 - 3abc = 0$ by Cardon's method.

Sol. Given equation is $x^3 + 3ax^2 + 3(a^2 - bc)x + a^3 + b^3 + c^3 - 3abc = 0$

Here $a_0 = 1$, $a_1 = 3a$

$\therefore$ $$h = -\frac{a_1}{n.a_0} = -\frac{3a}{3.1} = -a$$

Now remove the x^2 term, using synthetic division, we have

$-a$	1	$3a$	$3(a^2 - bc)$	$a^3 + b^3 + c^3 - 3abc$
		$-a$	$-2a^2$	$-a^3 \quad + 3abc$
	1	$2a$	$a^2 - 3bc$	$(b^3 + c^3)$
		$-a$	$-a^2$	
	1	a	$-3bc$	
		$-a$		
	1	(0)		