Chapter 6

Taylor and Laurent Expansions—Analytic Continuation

6.1 Taylor expansion

Let $f(z)$ be analytic within and on a circle C with center at z_0. Let z be a point *within* the circle. Then Cauchy's integral formula can be written as

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(z')}{z'-z}dz' = \frac{1}{2\pi i}\oint_C \frac{f(z')}{(z'-z_0)-(z-z_0)}dz'. \tag{6.1}$$

Because z lies inside the circle,

$$|z'-z_0| > |z-z_0|, \tag{6.2}$$

we can expand the denominator,

$$\frac{1}{(z'-z_0)-(z-z_0)} = \frac{1}{z'-z_0}\frac{1}{1-\frac{z-z_0}{z'-z_0}} = \frac{1}{z'-z_0}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{z'-z_0}\right)^n. \tag{6.3}$$

This series converges absolutely and uniformly for z' on the circle and z fixed inside, so it may be integrated term by term:

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty}(z-z_0)^n \frac{1}{2\pi i}\oint_C \frac{f(z')}{(z'-z_0)^{n+1}}dz' \\ &= \sum_{n=0}^{\infty}(z-z_0)^n \frac{1}{n!}f^{(n)}(z_0), \end{aligned} \tag{6.4}$$

using the result of Eq. (5.34).

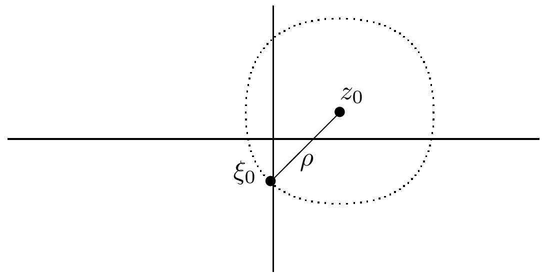

Figure 6.1: Circle of convergence for the Taylor series (6.4). Here z_0 is the point about which the Taylor expansion is performed, ξ_0 is the closest singularity of f to z_0, and $\rho = |\xi_0 - z_0|$ is the radius of convergence. The Taylor series converges within the circle of convergence, and diverges outside the circle of convergence. It may either diverge or converge on the circle of convergence.

This Taylor series will converge inside a circle having radius equal to the distance from z_0 to the nearest singularity, and diverge outside such a circle, as illustrated in Fig. 6.1.

Proof: For $|z - z_0| < \rho$, we can choose C in the above derivation to have radius r, where $|z-z_0| < r < \rho$, so the above expansion converges. For $|z-z_0| > \rho$, suppose it were true that the Taylor series converged. Then, according to the theorem in Sec. 2.7, it would converge at $z = \xi_0$, to an analytic function (Sec. 5.9). This is contrary to the assertion that ξ_0 is a singular point. QED.

Example

Consider the function

$$f(z) = \frac{1}{1-z}, \tag{6.5}$$

which is analytic except at $z = 1$. The Taylor series about the origin,

$$\frac{1}{1-z} = 1 + z + z^2 + \dots, \tag{6.6}$$

converges only for $|z| < 1$. We may obtain a larger circle of convergence by expanding about some other point, say $z = -1$:

$$\begin{aligned}\frac{1}{1-z} &= \frac{1}{2-(z+1)} = \frac{1}{2}\frac{1}{1-\frac{z+1}{2}} \\ &= \frac{1}{2}\left[1 + \frac{z+1}{2} + \left(\frac{z+1}{2}\right)^2 + \dots\right],\end{aligned} \tag{6.7}$$

which converges inside a circle of radius 2, centered about $z = -1$. In both cases the singularity at $z = 1$ lies on the circle of convergence.

6.2 Analytic Continuation

The process of *extending* a power series representation of an analytic function is called *analytic continuation.* It can be done whenever there are only isolated singular points. The general idea is as follows.

Suppose we have a power series about z_0

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \tag{6.8}$$

which has radius of convergence ρ. (That is, it converges if $|z - z_0| < \rho$ and diverges if $|z - z_0| > \rho$.) The function f has a singular point somewhere on the circle of convergence. Since this power series represents an analytic function inside its circle of convergence, it can, by the above, be Taylor expanded about any other point lying within the circle of convergence, say z_1,

$$f(z) = \sum_{n=0}^{\infty} b_n (z - z_1)^n. \tag{6.9}$$

In general,[1] the circle of convergence of this series will lie partly outside the original circle. Thus f is now defined in a larger domain. In the new region, f may be expanded once again, and usually the new circle of convergence will lie partly outside both the first two circles, so again the meaning is extended. And so on. The idea is sketched in Fig. 6.2.

Entire functions may be represented by power series (Taylor expansions) valid everywhere, since they have no singular points.

6.3 Laurent Expansion

Let $f(z)$ be analytic in the annulus defined by two concentric circles C_1 and C_2, both centered on z_0, including the bounding circles. See Fig. 6.3. If z lies in the annulus, Cauchy's integral formula says (the interior boundary C_1 must be traversed in a clockwise sense—hence, the minus sign)

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \oint_{C_2} \frac{f(z')}{z' - z} dz' - \frac{1}{2\pi i} \oint_{C_1} \frac{f(z')}{z' - z} dz' \\ &= \frac{1}{2\pi i} \oint_{C_2} \frac{f(z')\, dz'}{(z' - z_0) - (z - z_0)} - \frac{1}{2\pi i} \oint_{C_1} \frac{f(z')\, dz'}{(z' - z_0) - (z - z_0)}. \end{aligned} \tag{6.10}$$

For the C_2 integral, $|z - z_0| < |z' - z_0|$ so we expand in $(z - z_0)/(z' - z_0)$; for the C_1 integral $|z - z_0| > |z' - z_0|$, so we expand in $(z' - z_0)/(z - z_0)$. Thus we have

$$f(z) = \frac{1}{2\pi i} \oint_{C_2} f(z') \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(z' - z_0)^{n+1}} dz'$$

[1] But not always. See Whittaker and Watson, §5.501.

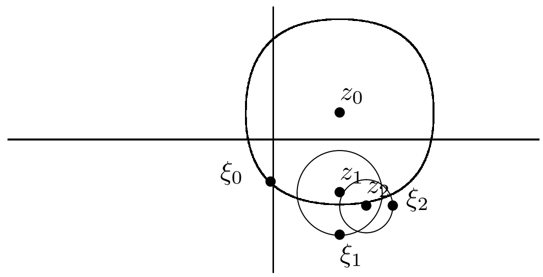

Figure 6.2: The process of analytic continuation of a function defined by a power series. The original series is a Taylor expansion about the point z_0, which converges inside a circle having radius equal to the distance to the nearest singularity ξ_0. If the function is instead expanded about the point z_1, it converges in a different circle, having radius equal to the distance from z_1 to the singular point closest to z_1, namely ξ_1. Instead the function can be expanded about z_2, lying outside the first circle of convergence, but inside the second, which will define the function in a different circle of convergence, with radius of convergence equal to the distance to the singularity closest to z_2, namely, ξ_2. This process may be repeated indefinitely. f is defined in the union of all such circles of convergence.

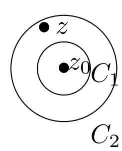

Figure 6.3: Annular region defined by two concentric circles.

$$+\frac{1}{2\pi i}\oint_{C_1} f(z')\sum_{n=0}^{\infty}\frac{(z'-z_0)^n}{(z-z_0)^{n+1}}dz'. \tag{6.11}$$

Now $\oint_C f(z')(z'-z_0)^k dz'$, where k is a positive or negative integer, has the same value for all contours circling z_0 once and lying in the annulus, since $f(z')(z'-z_0)^k$ is analytic there. Therefore the two sums above may be combined to yield

$$f(z)=\sum_{n=-\infty}^{\infty}a_n(z-z_0)^n, \tag{6.12}$$

where the expansion coefficients are

$$a_n=\frac{1}{2\pi i}\oint_C \frac{f(z')}{(z'-z_0)^{n+1}}dz'. \tag{6.13}$$

where C is any contour lying in the annulus. This is called the *Laurent expansion.* It generalizes the Taylor expansion in the case when there are singularities interior to C_1. (When there are no such singularities, the terms for negative n are identically zero.)

Example

The function

$$\exp\left[\frac{x}{2}\left(z-\frac{1}{z}\right)\right] \tag{6.14}$$

is analytic except at $z=0$. So it has a Laurent expansion about zero:

$$\exp\left[\frac{x}{2}\left(z-\frac{1}{z}\right)\right]=\sum_{n=-\infty}^{\infty}a_n z^n, \tag{6.15}$$

where

$$a_n=\frac{1}{2\pi i}\oint_C e^{\frac{x}{2}\left(z'-\frac{1}{z'}\right)}\frac{dz'}{z'^{\,n+1}}. \tag{6.16}$$

We make this last integral more explicit by choosing C to be a circle of unit radius, $z'=e^{i\theta}$, so

$$\begin{aligned} a_n &= \frac{1}{2\pi i}\int_0^{2\pi} e^{ix\sin\theta}e^{-in\theta}i\,d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi}\cos\left(n\theta-x\sin\theta\right)d\theta, \end{aligned} \tag{6.17}$$

because

$$\int_0^{2\pi}\sin\left(n\theta-x\sin\theta\right)d\theta=0, \tag{6.18}$$

owing to the integrand changing sign under the substitution $\theta \to 2\pi - \theta$. This function

$$\exp\left[\frac{x}{2}\left(z - \frac{1}{z}\right)\right] = \sum_{n=-\infty}^{\infty} z^n J_n(x) \tag{6.19}$$

is the *generating function* for the Bessel functions of integer order, $J_n(x)$. Thus we have derived the following integral representation of the Bessel functions,

$$J_n(x) = \frac{1}{2\pi}\int_0^{2\pi} \cos\left(n\theta - x\sin\theta\right) d\theta. \tag{6.20}$$

6.3.1 Example

Here is another example, which shows that the Laurent expansion holds for functions with branch points and branch lines, provided those are entirely inside the inner annular boundary. Consider the function

$$\sqrt{z^2 - 1}. \tag{6.21}$$

This function has branch points at $z = +1$ and at $z = -1$, and a branch line connecting these two points. Because it is a square-root singularity, the branch line for $|z| > 1$ cancels, as may be seen by considering the net phase change when *both* branch points are encircled:

$$\arg\left(\sqrt{z^2-1}\right)\Big|_{\arg z=0}^{\arg z=2\pi} = 0 \mod 2\pi. \tag{6.22}$$

This means that we may take $\sqrt{z^2} = z$, and we can immediately write down the expansion for large z:

$$\begin{aligned}\sqrt{z^2-1} &= z\left(1 - \frac{1}{z^2}\right)^{1/2} \\ &= z\left(1 - \frac{1}{2z^2} - \frac{1}{8z^4} - \dots\right) \\ &= -\sum_{n=0}^{\infty} \frac{(2n-3)!!}{2^n n!}\frac{1}{z^{2n-1}},\end{aligned} \tag{6.23}$$

where we have used the double factorial notation,

$$(2k+1)!! = (2k+1)(2k-1)(2k-3)\cdots 3\cdot 1, \tag{6.24}$$

and, from the recursion formula

$$(2k+1)!! = (2k+1)(2k-1)!! \tag{6.25}$$

identify

$$(-1)!! = 1, \quad (-3)!! = -1. \tag{6.26}$$

The Laurent expansion (6.23) converges for $|z| > 1$.

6.4 Classification of Singularities

Suppose in the neighborhood of z_0 a function $f(z)$ may be written as

$$f(z) = \phi(z) + \frac{a_{-1}}{z - z_0} + \frac{a_{-2}}{(z - z_0)^2} + \ldots + \frac{a_{-n}}{(z - z_0)^n}, \tag{6.27}$$

where $\phi(z)$ is analytic in the neighborhood of z_0, and a_{-1}, a_{-2}, …, a_{-n} are complex constants. When the above expansion holds true, f is said to have a *pole* of order n at $z = z_0$. When $n = 1$, the singularity is called a *simple pole.* When f has a pole of order n at z_0,

$$(z - z_0)^n f(z) \tag{6.28}$$

is analytic at $z = z_0$. If the function

$$(z - z_0)^m f(z) \tag{6.29}$$

is not analytic at $z = z_0$ no matter how large the integer m is, we say that f has an *essential singularity* at z_0. (This definition applies to functions which are single-valued without the introduction of branch lines.)

If an essential singularity is "isolated," that is, in a sufficiently small neighborhood of z_0, f is analytic except at z_0, f may be expanded in a Laurent series converging in an annulus:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n, \quad \Delta > |z - z_0| > \delta, \tag{6.30}$$

where δ is arbitrarily small, and Δ is the distance to the next singularity. (The proof for this statement is provided in the homework.)

6.4.1 Weierstrass–Picard Theorem

In the neighborhood of an essential singularity, $f(z)$ becomes arbitrarily close to every complex value. This theorem, due to Weierstrass, was greatly sharpened by Picard.

Picard's Theorem

In any neighborhood of an essential singularity, the function assumes every finite value, with one possible exception, an infinite number of times.

Example: Consider

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!\, z^n}, \tag{6.31}$$

which has an essential singular point at $z = 0$. Let α be any complex number except 0. For what zs is

$$\alpha = e^{1/z}? \tag{6.32}$$

Recalling the $2\pi i$ periodicity of the exponential function, we see

$$\log\alpha = \frac{1}{z} + 2\pi i n, \quad n = \text{integer}, \tag{6.33}$$

or

$$z = \frac{1}{\log\alpha - 2\pi i n}. \tag{6.34}$$

Thus in any neighborhood of 0 there are an infinite number of these zs.

6.4.2 Branch Points and Cuts

Recall $\log z$ was defined in the cut plane shown in Fig. 3.1. The location of the cut line is arbitrary, but the location of the end point, $z = 0$ is not. This *branch point* is a singular point of $\log z$:

$$\frac{d}{dz}\log z = \frac{1}{z}, \tag{6.35}$$

which does not exist at $z = 0$. This type of singularity is neither a pole nor an essential singularity. Once the cut is specified, thus defining $\log z$, the function is not analytic on the *branch cut* or *branch line*; in fact, it is discontinuous across the cut:

$$\operatorname{disc}(\log z) = \log \rho e^{i\pi} - \log \rho e^{-i\pi} = 2i\pi. \tag{6.36}$$

The same applies to square roots, and all nonintegral powers, which are defined in terms of the logarithm,

$$\sqrt{z} = z^{1/2} = e^{\frac{1}{2}\log z}. \tag{6.37}$$

Here the discontinuity across the branch line is

$$\begin{aligned}\operatorname{disc}(\sqrt{z}) &= \sqrt{\rho e^{i\pi}} - \sqrt{\rho e^{-i\pi}} \\ &= \sqrt{\rho}\left(e^{i\pi/2} - e^{-i\pi/2}\right) = 2i\sqrt{\rho}.\end{aligned} \tag{6.38}$$

6.5 Liouville's Theorem

First we prove *Cauchy's inequality.* Recall the integral representation for the derivative of an analytic function, Eq. (5.34),

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz \tag{6.39}$$

if z_0 is inside C and f is analytic on and within C. If C is a circle of radius r centered about z_0,

$$z = z_0 + re^{i\theta}, \tag{6.40}$$

we write this integral more explicitly as

$$f^{(n)}(z_0) = \frac{n!}{2\pi}\int_0^{2\pi} \frac{f\left(z_0 + re^{i\theta}\right)}{r^n e^{in\theta}} d\theta \tag{6.41}$$

or

$$\left|f^{(n)}(z_0)\right| \leq \frac{n!}{2\pi} \int_0^{2\pi} \frac{\left|f\left(z_0 + re^{i\theta}\right)\right|}{r^n} d\theta \leq \frac{n!M}{r^n}, \tag{6.42}$$

where M is the maximum value attained by $|f|$ on C.

Now *Liouville's theorem* (also really due to Cauchy) states: *An entire bounded function is constant.*

Proof: Since $f(z)$ is entire, the Taylor series converges everywhere,

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0) z^n. \tag{6.43}$$

But from Cauchy's inequality,

$$\left|f^{(n)}(0)\right| \leq \frac{Mn!}{R^n}, \tag{6.44}$$

where R is the radius of an arbitrarily large circle about the origin, and M may be taken as the bound on $|f|$,

$$|f(z)| \leq M \quad \forall z. \tag{6.45}$$

Hence by taking $R \to \infty$, we see that

$$f^{(n)}(0) = 0, \quad n > 0, \tag{6.46}$$

and so

$$f(z) = f(0). \tag{6.47}$$

QED.

Example:

Although e^z is entire, is is certainly not bounded.