Class 2/14

1 Singularities

Definition 1. f has an isolated singularity at $z = a$ if there is a punctured disk $B(a, R)\backslash\{a\}$ such that f is defined and analytic on this set, but not on the full disk. a is called removable singularity if there is an analytic $g : B(a, R) \to \mathbb{C}$ such that $g(z) = f(z)$ for $0 < |z - a| < R$.

Investigation of removable singularities.

Example 1. $f(z) = z^{-1} \sin z$ has a removable singularity at the origin. To see this, divide the power expansion of $\sin z$ by z.

Theorem 1. *If f has an isolated singularity at a, then the point $z = a$ is a removable singularity if and only if*

$$\lim_{z \to a}(z - a)f(z) = 0.$$

Proof. Need to show $\Longleftarrow$:

Suppose that f is analytic for $0 < |z-a| < R$. Define $g(z) = (z-a)f(z)$ for $z \neq a$ and $g(a) = 0$, and note that by assumption g is continuous at a. We will show that g is analytic in $B(a, R)$. Since an analytic function has zeros of integer order, it then follows that f is analytic in this disk.

We apply Morera's theorem. Let T be a triangular path in $B(a, R)$, enclosing the triangle Δ. If $a \notin \Delta$, then $\int_T g = 0$.

If a is a vertex of Δ, partition Δ into two triangles, one a very small one with vertex a, and use continuity of g on the smaller to bound the contribution to the integral by ε.

If $a \in \Delta$ is not a vertex, partition Δ into two or three triangles, and apply the previous case to each. □

Definition 2. The point a is called a pole of $f(z)$ if $\lim_{z\to a} |f(z)| = \infty$. If a is a singularity which is neither a pole, nor removable, then a is called an essential singularity of f.

The point $z = 0$ is an essential singularity of $e^{z^{-1}}$. (Consider different directional limits at the origin.)

Proposition 1. *Let G be a region with $a \in G$ and let f be analytic on $G \backslash \{a\}$. If f has a pole at a, then there exists an integer m and an analytic function g on G so that*

$$f(z) = \frac{g(z)}{(z-a)^m}.$$

Proof. Consider $\phi = 1/f$. By assumption, $\lim_{z \to a} \phi(z) = 0$, hence ϕ can be continued as an analytic function to G by setting $\phi(a) = 0$. (A priori ϕ is continuous at a and analytic on $G \backslash \{a\}$, so apply Morera's theorem as in the previous proof.)

Hence can find m and γ so that $\phi(z) = (z-a)^m \gamma(z)$ with $\gamma(a) \neq 0$. Define $g = 1/\gamma$. □

In the situation above, a is called a pole of order m of $f(z)$. (Note also that $g(a) \neq 0$ since the same is true for γ.) Starting from the power series expansion of g about a gives

$$f(z) = \sum_{k=1}^{m} A_k (z-a)^{-k} + g_1(z), \tag{1}$$

where g_1 is analytic in a disk about a and $A_m \neq 0$. The first sum is called the singular part of f at a.

Consider this for reduced rational functions. Poles are then zeros of the denominator.