Evaluating Integrals Using Self-Similarity

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In memory of Henrietta Mazen

1. INTRODUCTION. It is generally believed that there are two elementary methods for evaluating definite integrals:

1. the method of Archimedes, in which you partition the interval, form an approximating sum (often called a Riemann sum), and take a limit;
2. the method of Newton and Leibniz, involving the Fundamental Theorem of the Calculus, in which you find an antiderivative by inspired guesswork, and then compute the increment over the interval.

There are more advanced methods, especially Cauchy's residue calculus using complex variable theory, but a typical calculus course mentions only these two methods. Students quickly learn from experience that Archimedes really was a genius, but for practical purposes the Fundamental Theorem is the way to go.

But there is a third method, which is quite elementary although not well known. It is derived from the theory of integration on fractals, and is based on a self-similarity property of the unit interval. It is not a truly practical method, since it gives exact answers only for integrals of polynomials, but it illustrates the important mathematical article of faith that the study of exotic structures can produce new insights into old and commonplace subjects. After presenting the method in the context of ordinary integrals, I indicate how it can be adapted to the context of integrals on fractals, where it is essentially the only method available.

It is necessary to assume that the integral has a few elementary properties:

(i) linearity,

$$\int_a^b (c_1 f_1(x) + c_2 f_2(x))\, dx = c_1 \int_a^b f_1(x)\, dx + c_2 \int_a^b f_2(x)\, dx$$

(ii) additivity,

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$$

(iii) linear change of variable,

$$c \int_a^b f(cx + d)\, dx = \int_{ca+d}^{cb+d} f(x)\, dx$$

(iv) integration of constants,

$$\int_a^b c\, dx = c(b - a).$$

These properties are easily derived from any reasonable definition of the integral. Of course (iii) is a special case of the more general change of variable formula, which is not as simple.

2. SELF-SIMILARITY OF THE INTERVAL. For simplicity of notation we work with the unit interval [0, 1], but the same considerations apply to any interval. If we break the interval in half, then each half is similar to the whole: the left half $[0, \frac{1}{2}]$

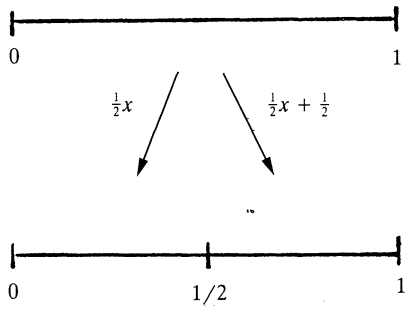

Figure 1

is the image of $[0, 1]$ under the mapping $\frac{1}{2}x$, while the right half $[\frac{1}{2}, 1]$ is the image of $[0, 1]$ under the mapping $\frac{1}{2}x + \frac{1}{2}$. Both mappings are similarities because they scale down all distances by a factor of $\frac{1}{2}$; see Figure 1.

For any function f, we have

$$\int_0^1 f(x)dx = \int_0^{1/2} f(x)\,dx + \int_{1/2}^1 f(x)\,dx$$

by additivity, while

$$\int_0^{1/2} f(x)\,dx = \frac{1}{2}\int_0^1 f\left(\frac{1}{2}x\right)dx$$

and

$$\int_{1/2}^1 f(x)\,dx = \frac{1}{2}\int_0^1 f\left(\frac{1}{2}x + \frac{1}{2}\right)dx$$

by the linear change of variable property. Altogether we obtain the *self–similar identity*

$$\int_0^1 f(x)\,dx = \frac{1}{2}\int_0^1 f\left(\frac{1}{2}x\right)dx + \frac{1}{2}\int_0^1 f\left(\frac{1}{2}x + \frac{1}{2}\right)dx. \tag{1}$$

There is a simple geometric explanation of this identity: the region under the graph of f is cut in half and each half is stretched horizontally by a factor of 2, which explains why we have to compensate by multiplying by the factor of $\frac{1}{2}$; see Figure 2.

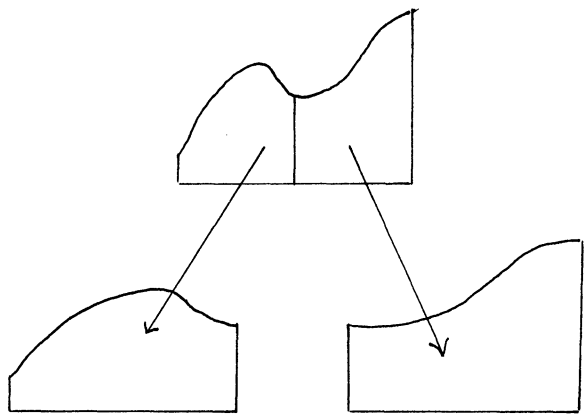

Figure 2

The self–similar identity expresses the consequences for the integral of the self–similarity of the interval. Aside from its intrinsic interest, it can be used as a tool for evaluating integrals. Before doing this, I want to mention a small point that the reader may have already noticed: there are many other ways to express the self–similarity of the interval. You can break up the interval into more than two intervals (they do not even have to be the same length), each similar to the whole interval. For each such decomposition there is a corresponding self–similar identity. For the calculations we are going to do, the single identity (1) suffices, so we leave the general form as an exercise for the reader.

3. INTEGRALS OF POLYNOMIALS. We want to obtain the basic integration formula

$$\int_0^1 x^n \, dx = \frac{1}{n+1}, \quad n = 0, 1, 2, \ldots \tag{2}$$

as a consequence of the self–similar identity (1). We observe that (2) and properties (ii) and (iii) imply the more general integral formula

$$\int_a^b x^n \, dx = \frac{b^{n+1} - a^{n+1}}{n+1},$$

and we can integrate any polynomial using (i). Note that for $n = 0$, (2) is just our assumption (iv), so the first non–trivial case is $n = 1$. We can obtain this by taking $f(x) = x$ in (1). Then we have

$$\int_0^1 x \, dx = \frac{1}{2}\int_0^1 \left(\frac{1}{2}x\right) dx + \frac{1}{2}\int_0^1 \left(\frac{1}{2}x + \frac{1}{2}\right) dx,$$

and using linearity

$$\int_0^1 x \, dx = \frac{1}{2}\int_0^1 x \, dx + \frac{1}{4}\int_0^1 dx.$$

Note that $\int_0^1 x \, dx$ appears on both sides of the equation, but multiplied by different constants, and $\int_0^1 dx = 1$ by the $n = 0$ case. Thus we may bring all the $\int_0^1 x \, dx$ terms to the left side to obtain

$$\frac{1}{2}\int_0^1 x \, dx = \frac{1}{4}\int_0^1 dx = \frac{1}{4}$$

which implies (2) for $n = 1$. While this may not seem like such a big deal (we could actually evaluate this integral by interpreting it as the area of a triangle), you should notice that the answer $\frac{1}{2}$ was obtained as $(1/4)/(1 - 1/2)$, and this is different from the computation that occurs if you use method 1) or 2).

So what happens if you substitute $f(x) = x^2$ in (1)? You get

$$\begin{aligned}\int_0^1 x^2 \, dx &= \frac{1}{2}\int_0^1 \left(\frac{1}{2}x\right)^2 dx + \frac{1}{2}\int \left(\frac{1}{2}x + \frac{1}{2}\right)^2 dx \\ &= \frac{1}{8}\int_0^1 x^2 \, dx + \frac{1}{8}\int_0^1 x^2 \, dx + \frac{1}{4}\int_0^1 x \, dx + \frac{1}{8}\int_0^1 dx.\end{aligned}$$

Figure 3 shows the decomposition of the region under the graph of $f(x) = x^2$ corresponding to this identity. Bring all the $\int_0^1 x^2 \, dx$ terms to the left side, and use

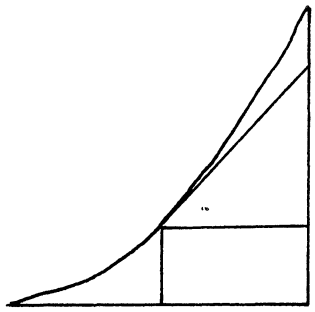

Figure 3

the previous evaluations ($n = 0$ or 1) to obtain

$$\frac{3}{4}\int_0^1 x^2\,dx = \frac{1}{4}\cdot\frac{1}{2} + \frac{1}{8}\cdot 1 = \frac{1}{4},$$

so the correct answer $\frac{1}{3}$ emerges as $(1/4)/(1 - 1/4)$.

Shall we try $n = 3$?

$$\int_0^1 x^3\,dx = \frac{1}{2}\int_0^1 \left(\frac{1}{2}x\right)^3 dx + \frac{1}{2}\int_0^1 \left(\frac{1}{2}x + \frac{1}{2}\right)^3 dx$$

$$= \frac{1}{16}\int_0^1 x^3\,dx + \frac{1}{16}\int_0^1 x^3\,dx + \frac{3}{16}\int_0^1 x^2\,dx + \frac{3}{16}\int_0^1 x\,dx + \frac{1}{16}\int_0^1 dx$$

hence

$$\frac{7}{8}\int_0^1 x^3\,dx = \frac{3}{16}\cdot\frac{1}{3} + \frac{3}{16}\cdot\frac{1}{2} + \frac{1}{16}\cdot 1 = \frac{7}{32}.$$

It is clear that we can continue indefinitely computing $\int_0^1 x^n\,dx$ for higher values of n, at each stage using the results previously obtained, but it is not obvious that this effort can produce the correct answer. This requires a little bit of algebra, and of course the finite binomial formula.

Let's try to prove (2) by induction, assuming that it holds for all integers less than n. Substituting $f(x) = x^n$ in (1) yields

$$\int_0^1 x^n\,dx = \frac{1}{2}\int_0^1 \left(\frac{1}{2}x\right)^n dx + \frac{1}{2}\int_0^1 \left(\frac{1}{2}x + \frac{1}{2}\right)^n dx$$

$$= \frac{1}{2^{n+1}}\left(\int_0^1 x^n\,dx + \int_0^1 (x+1)^n\,dx\right)$$

$$= \frac{1}{2^{n+1}}\left(2\int_0^1 x^n\,dx + \sum_{k=0}^{n-1}\binom{n}{k}\int_0^1 x^k dx\right).$$

The induction hypothesis ensures that $\int_0^1 x^k\,dx = 1/(k+1)$, since $k \le n - 1$. We thus obtain

$$\left(\frac{2^n - 1}{2^n}\right)\int_0^1 x^n\,dx = \frac{1}{2^{n+1}}\sum_{k=0}^{n-1}\binom{n}{k}\left(\frac{1}{k+1}\right).$$

To complete the proof of (2) we need to verify only the following algebraic lemma:

Lemma. $\sum_{k=0}^{n-1}\binom{n}{k}\left(\frac{1}{k+1}\right)=\left(\frac{2}{n+1}\right)(2^n-1)$.

Proof: Note that

$$\binom{n}{k}\left(\frac{n+1}{k+1}\right)=\left(\frac{n!}{k!(n-k)!}\right)\left(\frac{n+1}{k+1}\right)=\frac{(n+1)!}{(k+1)!(n-k)!}=\binom{n+1}{k+1}$$

so that

$$(n+1)\sum_{k=0}^{n-1}\binom{n}{k}\left(\frac{1}{k+1}\right)=\sum_{k=0}^{n-1}\binom{n+1}{k+1}=\sum_{j=1}^{n}\binom{n+1}{j}.$$

Since $\sum_{j=0}^{n+1}\binom{n+1}{j}=2^{n+1}$ and we are missing the two terms $j=0$ and $j=n+1$ that each contribute 1, we have $\sum_{j=1}^{n}\binom{n+1}{j}=2^{n+1}-2$.

4. ITERATION, AND THE CONNECTION WITH RIEMANN SUMS. There do not seem to be any other integrals that can be evaluated exactly by the self–similarity method, but we can use it to gain some insight into the integral of a general function. The self–similar identity (1), like many other aspects of self–similarity, seems to invite iteration. This just means that we substitute the same identity, for the function $f(\frac{1}{2}x)$ and $f(\frac{1}{2}x+\frac{1}{2})$, into the right side of (1). This yields

$$\int_0^1 f(x)\,dx=\frac{1}{4}\int_0^1 f\left(\frac{1}{4}x\right)dx+\frac{1}{4}\int_0^1 f\left(\frac{1}{4}x+\frac{1}{4}\right)dx+\frac{1}{4}\int_0^1 f\left(\frac{1}{4}x+\frac{2}{4}\right)dx$$
$$+\frac{1}{4}\int_0^1 f\left(\frac{1}{4}x+\frac{3}{4}\right)dx.$$

If we repeat the process again and again, we arrive at the expression

$$\int_0^1 f(x)\,dx=\frac{1}{2^m}\sum_{k=0}^{2^m-1}\int_0^1 f\left(\frac{1}{2^m}(x+k)\right)dx \tag{3}$$

for any positive integer m.

This resembles a Riemann sum. If we subdivide the interval into 2^m subintervals of length $1/2^m$ each, then the Riemann sums take the form

$$\frac{1}{2^m}\sum_{k=0}^{2^m-1}f(x_k)$$

where x_k lies in the interval $[k2^{-m},(k+1)2^{-m}]$. So the only difference is that instead of choosing a point x_k in the interval to evaluate the function at, we take the average value $\int_0^1 f((x+k)/2^m)dx$ of f over the interval. In contrast to the Riemman sum, we get the exact value of the integral at each stage, so it is not necessary to take the limit as $m\to\infty$. But we certainly are allowed to take this limit, and if we don't know how to evaluate the integrals on the right side of (3), then this is the best thing to do. If the function f is assumed to be continuous on [0, 1], then the values on each of the subintervals do not vary by very much for large values of m, so the average value and any typical value $f(x_k)$ are very close; strictly speaking, this argument requires uniform continuity, and the non–elementary theorem that a continuous function in [0, 1] is automatically uniformly continuous. The formula

$$\int_0^1 f(x)\,dx=\lim_{m\to\infty}\frac{1}{2^m}\sum_{k=0}^{2^m-1}\int_0^1 f\left(\frac{1}{2^m}(x+k)\right)dx$$

is almost the same as the usual definition of the integral as the limit of Riemann sums. This is just method 1) for evaluating integrals, nothing really new. However, the Riemann sums came out of the self–similar method. Also, the evaluation of integrals of polynomials in Section 3 did not use iteration and limits, and is genuinely different.

5. EXOTIC INTEGRALS ON THE INTERVAL. It is not necessary to pass to exotic fractal geometry to obtain exotic integrals. We can stay with the plain old interval, but consider exotic averages in place of the usual integral. The technical term is *integration with respect to self–similar measures*, but we can explain the ideas on an intuitive level by referring to probability. It is best to put aside the area interpretation of the integral, and think only about the average interpretation: $\int_0^1 f(x)\,dx$ is the average value of f on the interval. We also say that it is the *expected value* of f with respect to the uniform probability distribution on the interval. If we choose numbers x in $[0, 1]$ at random, then the integral gives the average value of $f(x)$ that we find.

How do we choose a number at random? Let's represent x in binary notation $x = .x_1x_2x_3\ldots$. We may ignore the ambiguity in the binary representation of certain numbers, such as $.1000\ldots = .0111\ldots$ because the set of all such numbers has zero probability. We choose $x_1 = 0$ or 1 with equal probability; then, independently, we choose $x_2 = 0$ or 1 with equal probability; and so on. Indeed, the self–similar identity (1) expresses this fact succinctly. Note that $\frac{1}{2}x = .0x_1x_2\ldots$ and $\frac{1}{2}x + \frac{1}{2} = .1x_1x_2\ldots$, and the factors $\frac{1}{2}$, $\frac{1}{2}$ on the right side of (1) are the equal probabilities of choosing 0 or 1 for the first digit. On the left side of (1) we have the average value of $f(x)$ over the interval, while on the right we have the sum of the probability of picking 0 or 1 for the first digit multiplied by the average value of $f(x)$ given that the first digit is 0 or 1.

But there are other ways to pick a number at random. A simple variant of the uniform distribution is to pick 0 and 1 with probabilities p and $1 - p$, for any fixed p in $(0, 1)$. This is called a *Bernoulli distribution*, and we write μ_p for the associated probability, so $\mu_p(A)$ is the probability that x belongs to a subset A of $[0, 1]$ when x is chosen in this manner. We write $\int f\,d\mu_p$ for the average value of f (or the expected value of f) when x is so chosen. When $p = 1/2$ this is the usual integral. In general it is a new kind of exotic integral. It satisfies properties (i), (ii), and (iv) of the integral, but not (iii). In place of (1), we have the self–similar identity

$$\int f(x)\,d\mu_p = p\int f\left(\frac{1}{2}x\right)d\mu_p + (1-p)\int f\left(\frac{1}{2}x + \frac{1}{2}\right)d\mu_p. \tag{4}$$

The explanation for (4) is the same as the explanation for (1) just given, with the difference that p and $(1 - p)$ are the respective probabilities for the first digit being 0 or 1.

There is no fundamental theorem of calculus for this integral. As far as I know, no one knows how to compute $\int e^x\,d\mu_p$ or $\int \sin x\,d\mu_p$ or $\int \sqrt{x}\,d\mu_p$. But we can compute the integral of polynomials, using the self–similar identity as before. We start with $\int 1\,d\mu_p = 1$, as this is a property of all averages. Then

$$\int x\,d\mu_p = p\int\left(\frac{1}{2}x\right)d\mu_p + (1-p)\int\left(\frac{1}{2}x + \frac{1}{2}\right)d\mu_p = \frac{1}{2}\int x\,d\mu_p + \frac{1-p}{2}\int 1\,d\mu_p$$

hence

$$\frac{1}{2}\int x\,d\mu_p = \frac{1-p}{2}$$

or

$$\int x\, d\mu_p = 1 - p.$$

By the way, notice that we also have

$$\int (1 - x)\, d\mu_p = p.$$

A heuristic explanation for these results is that if $p > \frac{1}{2}$ then the probability gives more weight to the interval $[0, \frac{1}{2}]$, where x takes smaller values and $(1 - x)$ takes larger values, so $\int x\, d\mu_p$ should be less than $\frac{1}{2}$. Next

$$\begin{aligned}\int x^2\, d\mu_p &= p\int \left(\frac{1}{2}x\right)^2 d\mu_p + (1-p)\int \left(\frac{1}{2}x + \frac{1}{2}\right)^2 d\mu_p \\ &= \frac{1}{4}\int x^2\, d\mu_p + (1-p)\left(\frac{1}{2}\int x\, d\mu_p + \frac{1}{4}\int 1\, d\mu_p\right) \\ &= \frac{1}{4}\int x^2\, d\mu_p + (1-p)\left(\frac{1}{2}(1-p) + \frac{1}{4}\right)\end{aligned}$$

so

$$\int x^2\, d\mu_p = \frac{4}{3}(1-p)\left(\frac{1}{2}(1-p) + \frac{1}{4}\right) = \frac{2}{3}(1-p)^2 + \frac{1}{3}(1-p).$$

We can continue the process to higher powers, but the expressions become more and more complicated. It is easy to show that

$$\int x^n\, d\mu_p = \sum_{j=1}^{n} A_{nj}(1-p)^j$$

for positive coefficients A_{nj}, where $A_{n1} = (2^n - 1)^{-1}$, and the recursion relations

$$A_{nj} = \frac{1}{2^n - 1}\sum_{k=j-1}^{n-1} \binom{n}{k} A_{k(j-1)}$$

hold for $2 \le j \le n$. However, there does not appear to be any explicit formula for the general coefficient.

6. POLYGASKETS AND OTHER FRACTALS. The familiar Sierpinski gasket SG (Figure 4) is one of the simplest examples of a self–similar set that is truly fractal. Just as in the case of the interval, there are subsets of SG that are similar to the whole. In particular, SG is the union of three similar gaskets of exactly half the size. We can write a self–similar identity for the set SG in the form $K = F_1K \cup F_2K \cup F_3K$ where F_j denotes the contractive similarity $F_jx = \frac{1}{2}(x - q_j) + q_j$ in the plane with contraction ratio $\frac{1}{2}$ and fixed point q_j (so $x = (x_1, x_2)$ denotes a point in the plane), where q_1, q_2, q_3 are the vertices of an equilateral triangle. Here the variable K denotes a closed, bounded, non–empty set in the plane. It can be shown that $K = SG$ is the only solution of this identity. More generally, we can write self–similar identities

$$K = \bigcup_{j=1}^{n} F_jK$$

for any finite family of contractive similarities, called an *iterated function system*.

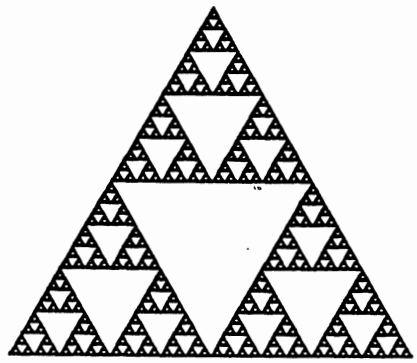

Figure 4

The unique solution is called a self–similar set. We concentrate on a family of examples called *polygaskets*, obtained as follows. We take the n vertices $q_1, q_2, \ldots, q_n$ of a regular n–gon, and the contractive similarities $F_j x = r_n(x - q_j) + q_j$ with fixed–points q_j. We choose the contraction ratio r_n so that the n images of the original n–gon just touch. It is an exercise in geometry to show that the correct value is

$$r_n = \frac{\sin \pi/n}{\sin \pi/n + \sin(\pi/n + 2\pi m/n)} \quad \text{for} \quad m = [n/4].$$

Figure 5 shows the pentagasket, hexagasket, and octagasket, corresponding to $n = 5, 6$, and 8. Note that the octagons touch along edges rather than at vertices;

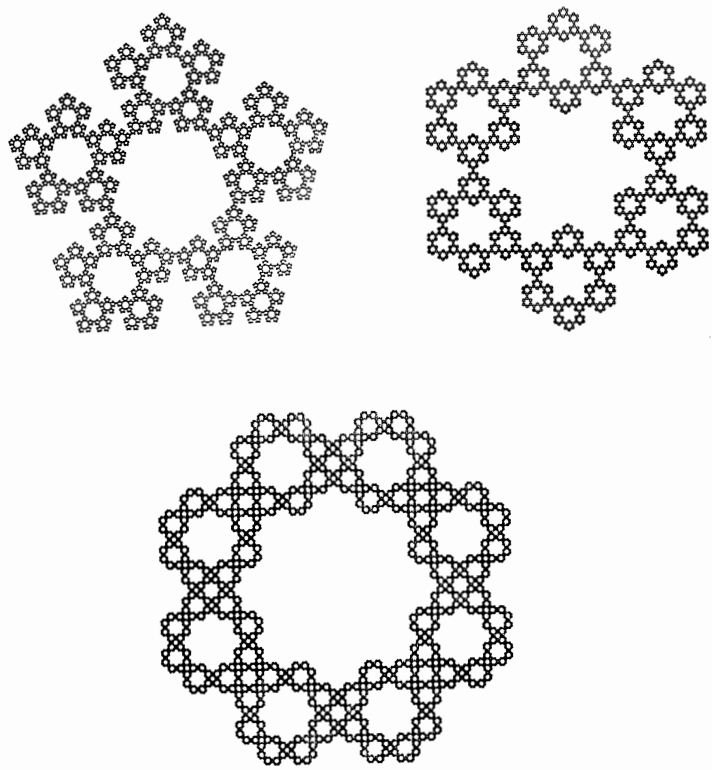

Figure 5

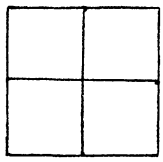

Figure 6

this happens exactly when n is divisible by 4. When $n = 2$ we get the interval, and when $n = 4$ we get the square, as Figure 6 shows.

We want to consider probability measures μ on these fractals, and the associated integrals $\int f d\mu$ for functions f defined on the fractal. Since the fractals are embedded in the plane we can imagine that f is just given by restricting to K a function defined on the plane, but it is not necessary that f be defined at points not in K. The intuition is that $\int f d\mu$ represents the average value of $f(x)$ when x is chosen at random from K according to the probability distribution. Since K is self-similar, we would like the probability distribution to be self-similar as well, so that the probability of choosing a point in a subset A of $F_j K$ is just proportional to the probability of choosing a point in the corresponding subset, which is $F_j^{-1}A$, in K. That is,

$$\mu(A) = p_j \mu\left(F_j^{-1}A\right) \quad \text{for} \quad A \subseteq F_j K.$$

Here we need to make an important assumption: the overlaps $F_j K \cap F_k K$ for $j \neq k$ have zero probability. This is true for the polygaskets because these are either points (n not divisible by 4) or subsets of line segments (n divisible by 4), but it is not true in general. With this assumption we see that the constants of proportionality p_j must form a set of finite probabilities, so $0 < p_j < 1$ and $p_1 + \cdots + p_n = 1$; to avoid degenerate cases we require $p_j > 0$. Also, we can combine all n equations into one:

$$\mu(A) = \sum_{j=1}^{n} p_j \mu\left(F_j^{-1}A\right) \quad \text{for} \quad A \subseteq F,$$

called a self-similar identity for the probability measure μ. The corresponding identity for the integral is

$$\int f d\mu = \sum_{j=1}^{n} p_j \int f \circ F_j \, d\mu. \tag{5}$$

This is the analog of our original self-similar identity (1). Informally, this self-similar identity says that to find the average value of f on K, take the sum of the product of the probability p_j that a random point x belongs to $F_j K$ times the average value of f on $F_j K$; as x varies over K, $F_j x$ varies over $F_j K$, so $\int f \circ F_j d\mu$ represents this average.

We can iterate (5) just as we did (1), to obtain

$$\int f d\mu = \sum_{j_1=1}^{n} \cdots \sum_{j_m=1}^{n} p_{j_1} \cdots p_{j_m} \int f \circ F_{j_1} \circ \cdots \circ F_{j_m} \, d\mu. \tag{6}$$

We can again interpret this as a kind of Riemann sum. At each level m, we partition K into the n^m subsets $F_{j_1} \circ \cdots \circ F_{j_m} K$, and we assign the weight $p_{j_1} \cdots p_{j_m}$ to each subset. The weights play the role of the lengths of subintervals in a usual

Riemann sum. In (6) the weight is multiplied by the average value of f on the subset, but we could replace the average by a typical value of f at some point in the subset, at least for continuous functions f, by taking a limit as $m \to \infty$. In fact this is one way we can define the integral.

For the case of the polygaskets, the most natural choice is to take all the weights to be equal, $p_j = 1/n$ for all j. Then each subset $F_{j_1} \circ \cdots \circ F_{j_m} K$ at level m is a little n–gasket contracted by the ratio r_n^m, and given the probability weight $1/n^m$. Note that the ratio

$$d = \frac{\log(1/n^m)}{\log r_n^m} = \frac{\log(1/n)}{\log r_n} = \frac{\log n}{\log(1/r_n)}$$

is independent of m. We can interpret d as a dimension in the following sense: if we scale a figure by a factor r, we expect its weight to be scaled by the factor r^d, where d is its dimension. In fact this value of d coincides with all the usual dimensions for the polygasket (Hausdorff, box, packing, ...), and the self–similar probability measure with equal weights coincides, up to an unknown constant of proportionality, with the Hausdorff measure in dimension d.

Just as in the case of the interval, the self–similar identity can be used to evaluate the integrals of some functions on polygaskets, including any polynomial function (the restriction to K of a polynomial in 2 variables). The method is exactly the same. It uses the fact that $f \circ F_j$ is also a polynomial of the same degree, explicitly computable. Again it is necessary to proceed inductively on the degree of the polynomial, starting with $\int 1 d\mu = 1$. If f is the monomial $x_1^{k_1} x_2^{k_2}$, then $f \circ F_j$ is the sum of $r_n^{k_1+k_2} x_1^{k_1} x_2^{k_2}$ and terms of lower degree. Thus (5) yields

$$\int x_1^{k_1} x_2^{k_2} \, d\mu = \sum_{j=1}^{n} p_j r_n^{k_1+k_2} \int x_1^{k_1} x_2^{k_2} d\mu + I$$

where I consists of integrals already computed, and so

$$\int x_1^{k_1} x_2^{k_2} \, d\mu = \left(1 - r_n^{k_1+k_2}\right)^{-1} I.$$

We do not present the details. For general self–similar fractals we can also use essentially the same method. In cases when the similarities include rotations as well as homothetic contractions, it is necessary to do the computation simultaneously for all monomials of a fixed degree, which leads to a system of linear equations for the integrals.

This method of integration has been used for polynomial functions for a long time [7], and has been used for other, more intrinsic functions, in [4] and [6]. For more information on self–similar fractals, the reader may consult books about fractals, such as [1] or [2]. There is also a theory of differential calculus on fractals, but this is more intricate; see [3] or [5].

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From *The New Yorker*

The question, "Does mathematics need new axioms?," is ambiguous in practically every respect.

- What do we mean by "mathematics"?
- What do we mean by "need"?
- What do we mean by "axioms"?

You might even ask, What do we mean by "does"?—*American Mathematical Monthly*.

"New" apparently speaks for itself.

The New Yorker, May 10, 1999, p. 50

The quotation is the first paragraph of S. Feferman, Does Mathematics Need New Axioms?, MONTHLY 106 (1999) 99–111.