Fourier Series

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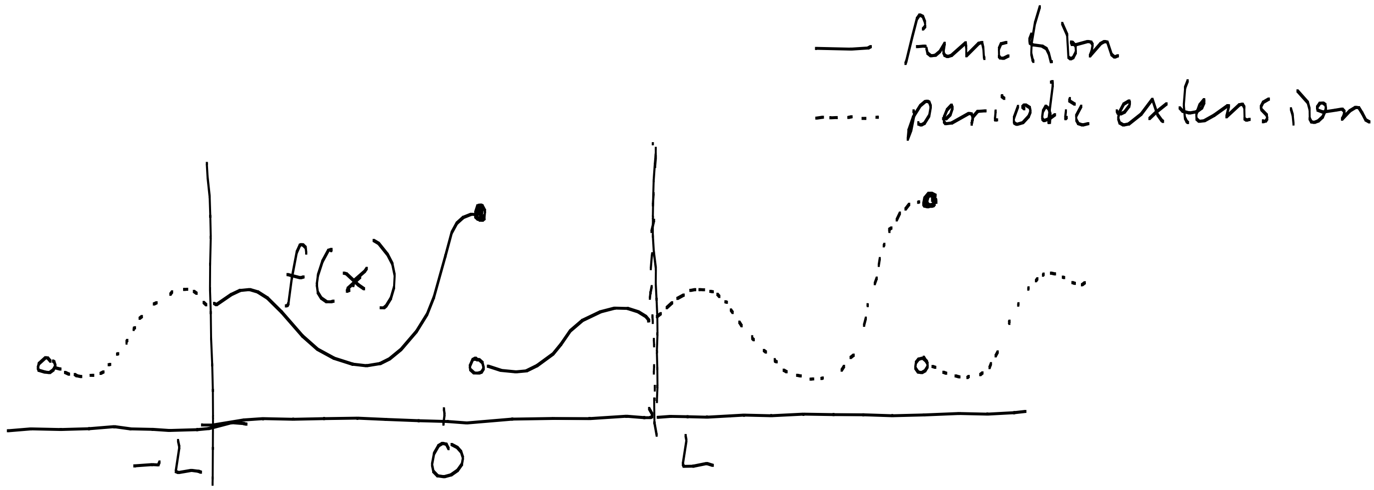


Dirichlet conditions for the existence of a
Fourier Series of a periodic function

① $f(x)$ is single valued with a finite number of discontinuities in $[-L, L]$

② $f(x)$ has a finite number of extrema in $[-L, L]$

③ $\int_0^L |f(x)|\,dx$ exists

"④ $f(x)$ is $2L$-periodic"

i.e. $f(x)$ belongs to the vector space spanned by

$$\cos\left[\frac{n\pi x}{L}\right]_{n=0,1,2,\ldots} \quad \text{and} \quad \sin\left[\frac{n\pi x}{L}\right]_{n=1,2,3,\ldots}$$

note the built-in periodicity:

$$\cos\left[\frac{n\pi x}{L}\right] = \cos\left[\frac{n\pi(x+2L)}{L}\right]$$

$$= \cos\left[\frac{n\pi x}{L} + 2n\pi\right]$$

$$\sin\left[\frac{n\pi x}{L}\right] = \sin\left[\frac{n\pi(x+2L)}{L}\right]$$

this means that any function built from a linear combination

$$g(x) = a_0 + \sum_{k=1}^{\infty} a_k \cos\left[\frac{k\pi x}{L}\right] + b_k \sin\left[\frac{k\pi x}{L}\right]$$

will also be periodic.

these basis functions are orthogonal for $m \neq n$

$2\sin x \sin y$
$= \cos(x-y) - \cos(x+y)$

$$\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{1}{2}\int_{-L}^{L}\left[\cos\left(\frac{(m-n)\pi x}{L}\right) - \cos\left(\frac{(m+n)\pi x}{L}\right)\right]dx$$

$[(m+n) > 0]$

0

$$= \frac{L}{2\pi}\left[\frac{1}{m-n}\sin\left(\frac{(m-n)\pi x}{L}\right) - \frac{1}{m+n}\sin\left(\frac{(m+n)\pi x}{L}\right)\right]\Bigg|_{-L}^{L}$$

$$= 0$$

$m - n \neq 0$

so if $m \neq n$, $\quad \langle \sin \frac{m\pi x}{L} | \sin \frac{n\pi x}{L} \rangle = 0$

and if $m = n$:

$$\langle \sin \frac{m\pi x}{L} | \sin \frac{m\pi x}{L} \rangle = \int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right)^2 dx$$

$$= \frac{1}{2}\int_{-L}^{L} \left[1 - \cos \frac{2\pi m x}{L}\right] dx$$

$$= \frac{1}{2}\left[x - \frac{\sin[2\pi m x/L]}{2\pi m/L}\right]\Bigg|_{-L}^{L}$$

$$= L$$

using ½ angle

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$$

<u>can also show that</u>:

$$\langle \cos \frac{m\pi x}{L} | \sin \frac{n\pi x}{L} \rangle = 0 \quad \forall\, m, n$$

$$\langle \cos \frac{m\pi x}{L} | \cos \frac{n\pi x}{L} \rangle = L\,\delta_{mn}$$

so the basis

$$\cos\left[\frac{n\pi x}{L}\right]_{n=0,1,2,\ldots} \quad \text{and} \quad \sin\left[\frac{n\pi x}{L}\right]_{n=1,2,3,\ldots}$$

is orthogonal (but not normalized) on $[-L, L]$

note that the case $m = 0$ gives

$$\langle \cos \frac{m\pi x}{L} | \cos \frac{m\pi x}{L} \rangle = 2L$$

an exception to our rule. it is customary to write the Fourier series in a way that auto-corrects for this:

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\frac{k\pi x}{L} + b_k \sin\frac{k\pi x}{L}$$

we can find the expansion coefficients using orthogonality:

$$\left\langle \cos\frac{m\pi x}{L} \middle| f(x) \right\rangle$$

$$= \int_{-L}^{L} \cos\frac{m\pi x}{L} f(x)\, dx$$

$$= \int_{-L}^{L} \cos\frac{m\pi x}{L} \left\{ \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\frac{k\pi x}{L} + b_k \sin\frac{k\pi x}{L} \right\} dx$$

$$= \begin{cases} \text{if } m=0 & \frac{1}{2}\int_{-L}^{L} a_0\, dx = L a_0 \\ \text{if } m \neq 0 & \int_{-L}^{L} a_m \left(\cos\frac{m\pi x}{L}\right)^2 dx = L a_m \end{cases}$$

b/c of $\frac{a_0}{2}$ convention

$$= L a_m$$

$$\left\langle \sin\frac{m\pi x}{L} \middle| f(x) \right\rangle =$$

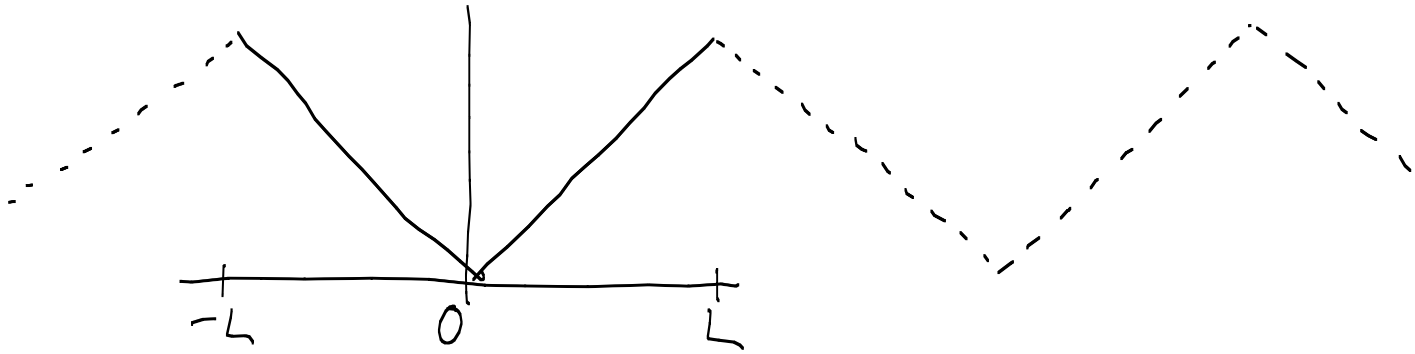
$$= \int_{-L}^{L} \sin\frac{m\pi x}{L} f(x)\, dx$$

$$= \int_{-L}^{L} \sin\frac{m\pi x}{L} \left\{ \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\frac{k\pi x}{L} + b_k \sin\frac{k\pi x}{L} \right\}$$

$$= \sum_{k=1}^{\infty} \int_{-L}^{L} \sin\frac{m\pi x}{L}\, b_k \sin\frac{k\pi x}{L}\, dx$$

$$= \sum_{k=1}^{\infty} b_k L \delta_{km} = L b_m$$

Example



$f(x) = |x|$ for $-L < x \leq L$

✓ ① $\int_{-L}^{L} |f(x)|\,dx$ exists

✓ ② $f(x)$ has finite extrema in $[-L, L]$

✓ ③ $f(x)$ has finite (0) discontinuities in $[-L, L]$

✓ ④ of course we can make periodic extension of $f(x)$ (as shown above)

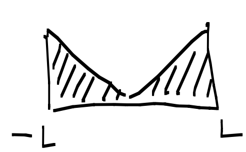
so there are $a_0, a_1, a_2, \dots$ and $b_1, b_2, \dots$ such that

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\frac{k\pi x}{L}\right) + b_k \sin\left(\frac{k\pi x}{L}\right)$$

what are the coefficients? [use formulas we just derived]

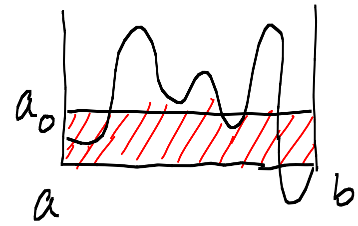
$m=0$:

$$L a_0 = \langle \cos \tfrac{0\pi x}{L} | f(x) \rangle$$

$$= \int_{-L}^{L} |x|\,dx =$$ 

$$= L^2 \Rightarrow \boxed{a_0 = L}$$

note that a_0 is the mean value of $f(x)$ on $[-L, L]$.



$m \geq 0$:

$$L a_m = \langle \cos \tfrac{m\pi x}{L} | f(x) \rangle$$

$$= \int_{-L}^{L} \cos \frac{m\pi x}{L} |x| \, dx$$

integrate by parts:
$u = x$ $dv = \cos$
[i used mma]

$$= 2 \int_0^L \cos \frac{m\pi x}{L} \, x \, dx$$

$$= 2 \left[-\frac{L^2}{m^2\pi^2} + \frac{L^2 \cos m\pi}{m^2\pi^2} + \frac{L^2 \sin m\pi}{m\pi} \right]$$

$$= 2L^2 \left[\frac{\cos m\pi - 1}{(m\pi)^2} + \cancel{\frac{\sin m\pi}{m\pi}} \right]$$

0 for $m \in \mathbb{Z}$

$$= \frac{2L^2}{(m\pi)^2} (\cos m\pi - 1)$$

$$= \begin{cases} -\dfrac{4L^2}{(m\pi)^2} & m = 1, 3, 5, \ldots \\ 0 & m = 2, 4, 6, \ldots \end{cases}$$

$$a_m = -\frac{4L}{(m\pi)^2}$$

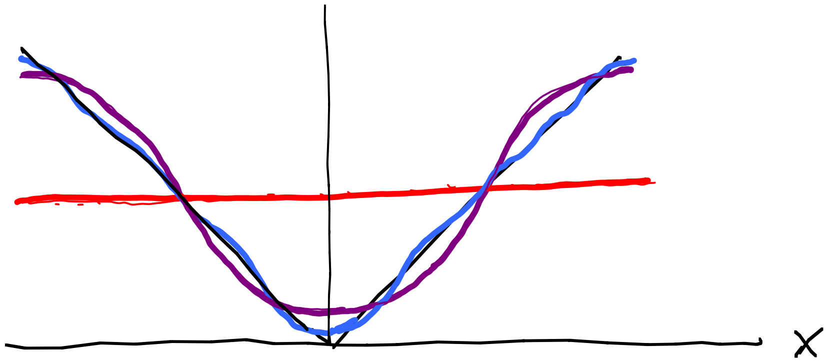
these should converge!

$$L b_m = \langle \underset{\text{"odd"}}{\sin \tfrac{m\pi x}{L}} | \underset{\text{"even"}}{f(x)} \rangle = 0$$

so we have:

$$f(x) = \frac{L}{2} - \frac{4L}{\pi^2} \cos \frac{\pi x}{L} - \frac{4L}{3^2\pi^2} \cos \frac{3\pi x}{L} - \frac{4L}{5^2\pi^2} \cos \frac{5\pi x}{L} \ldots$$

$$= \frac{L}{2} - \frac{4L}{\pi^2} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \cos\left((2k+1)\pi x / L\right)$$



$$\frac{L}{2}$$

$$\frac{L}{2} - \frac{4L}{\pi^2}\cos\frac{\pi x}{L}$$

$$\frac{L}{2} - \frac{4L}{\pi^2}\cos\frac{\pi x}{L} - \frac{4L}{3^2\pi^2}\cos\frac{3\pi x}{L}$$

$$\vdots$$

so why do we want to do this and what does it have to do with diagonalization?

$$\cos\left(\frac{k\pi x}{L}\right) \quad \text{and} \quad \sin\left(\frac{k\pi x}{L}\right)$$

are eigenfunctions of $\frac{d^2}{dx^2}$.

$$\frac{d^2}{dx^2}\cos\frac{k\pi x}{L} = -\left(\frac{k\pi}{L}\right)^2\cos\frac{k\pi x}{L}$$

$$\frac{d^2}{dk^2}\sin\frac{k\pi x}{L} = -\left(\frac{k\pi}{L}\right)^2\sin\frac{k\pi x}{L}$$

so we will use these eigenfunctions & eigenvalues to diagonalize (separate Fourier components) in partial differential equations like:

$$\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2} + f(x,t)$$

$$\Updownarrow \text{ compare to}$$

$$\dot{\vec{y}} = A\vec{y} + \vec{f}(t)$$

to "diagonalize" the PDE we will need homog. B.C.'s to obtain orthogonal eigenfunctions like the ones we used today.