# Lecture 2 Fourier Sine and Cosine Transformations

In this lecture we shall discuss the Fourier sine and cosine transforms and their properties. These transforms are appropriate for problems over semi-infinite intervals in a spatial variable in which the function or its derivative are prescribed on the boundary.

If a function is even or odd function then $f$ can be represented by a Fourier integral which takes a simpler form than in the case of an arbitrary function.

If $f(x)$ is an even function, then $B(\omega) = 0$ in (3), and

$$A(\omega) = 2\int_0^\infty f(t)\cos\omega t dt.$$

Hence, the Fourier integral reduces to the simpler form

$$f(x) = \frac{1}{\pi}\int_0^\infty A(\omega)\cos(\omega x) d\omega.$$

Similarly, if $f(x)$ is odd, then $A(\omega) = 0$ in (3), and

$$B(\omega) = 2\int_0^\infty f(t)\sin\omega t dt.$$

Thus, (3) becomes

$$f(x) = \frac{1}{\pi}\int_0^\infty B(\omega)\sin(\omega x) d\omega.$$

These Fourier integrals motivates to define the Fourier cosine transform (FCT) and Fourier sine transform (FST). The FT of an even function $f$ is called FCT of $f$. The FT of an odd function $f$ is called the FST of $f$.

**Definition 1. (Fourier Cosine Transform)** *The FCT of a function $f : [0,\infty) \to \mathbb{R}$ is defined as*

$$\mathcal{F}_c(f) = \hat{f}_c(\omega) = F_c(\omega) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos(\omega x) dx \quad (0 \le \omega < \infty). \tag{1}$$

**Definition 2. (Inverse Fourier Cosine Transform )** *The Inverse FCT (IFCT) of a function $\hat{f}_c(\omega)$ $(0 \le \omega < \infty)$ is defined as*

$$\mathcal{F}_c^{-1}[\hat{f}_c] = f_c(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_c(\omega)\cos(\omega x) d\omega \quad (0 \le x < \infty). \tag{2}$$

**Definition 3. (Fourier Sine Transform)** *The FST of a function $f : [0,\infty) \to \mathbb{R}$ is defined as*

$$\mathcal{F}_s(f) = \hat{f}_s(\omega) = F_s(\omega) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin(\omega x) dx \quad (0 \le \omega < \infty). \tag{3}$$

**Definition 4. (Inverse Fourier Sine Transform)** *The Inverse FST (IFST) of a function* $\hat{f}_s(\omega)$ $(0 \le \omega < \infty)$ *is defined as*

$$\boxed{\mathcal{F}_s^{-1}(f) = f_s(x) = F_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_s(\omega)\sin(\omega x)d\omega \quad (0 \le x < \infty).} \tag{4}$$

**Basic Properties of Fourier Cosine and Sine Transforms:**

- **Linearity:**

$$\begin{aligned} \mathcal{F}_c[(af+bg)] &= a\mathcal{F}_c[f] + b\mathcal{F}_c[g]. \\ \mathcal{F}_s[(af+bg)] &= a\mathcal{F}_s[f] + b\mathcal{F}_s[g]. \end{aligned}$$

- Let $f$ be a function defined for $x \ge 0$ and $f(x) \to 0$ as $x \to \infty$. Then

$$\begin{aligned} \mathcal{F}_s[f'(x)] &= \sqrt{\frac{2}{\pi}} \int_0^\infty \sin(\omega x) f'(x)dx \\ &= \sqrt{\frac{2}{\pi}} \sin(\omega x) f(x)\Big|_{x=0}^{x=\infty} - \omega\sqrt{\frac{2}{\pi}} \int_0^\infty \cos(\omega x) f(x)dx \\ &= -\omega \mathcal{F}_c[f]. \end{aligned}$$

If we assume that $f(x),\ f'(x) \to \infty$ then

$$\begin{aligned} \sqrt{\frac{2}{\pi}} \int_0^\infty \sin(\omega x) f''(x)dx &= \sqrt{\frac{2}{\pi}} \sin(\omega x) f'(x)\Big|_{x=0}^{x=\infty} - \omega\sqrt{\frac{2}{\pi}} \int_0^\infty \cos(\omega x) f'(x)dx \\ &= \sqrt{\frac{2}{\pi}} \sin(\omega x) f'(x)\Big|_{x=0}^{x=\infty} + \omega\sqrt{\frac{2}{\pi}} \cos(\omega x) f(x)\Big|_{x=0}^{x=\infty} \\ &\quad -\omega^2\sqrt{\frac{2}{\pi}} \int_0^\infty \sin(\omega x) f(x)dx \\ &= \omega\sqrt{\frac{2}{\pi}} f(0) - \omega^2 \mathcal{F}_s[f] \end{aligned}$$

Thus, we have

$$\begin{aligned} \mathcal{F}_s[f'(x)] &= -\omega \mathcal{F}_c[f]. \\ \mathcal{F}_s[f''(x)] &= -\omega^2 \mathcal{F}_s[f] + \omega\sqrt{\frac{2}{\pi}} f(0). \end{aligned}$$

A similar result is true for the Fourier cosine function.

$$\begin{aligned} \mathcal{F}_c[f'(x)] &= \omega \mathcal{F}_s[f] - \sqrt{\frac{2}{\pi}} f(0) \\ \mathcal{F}_c[f''(x)] &= -\omega^2 \mathcal{F}_c[f] - \sqrt{\frac{2}{\pi}} f'(0). \end{aligned}$$

**Note:** Observe that the FST of a first derivative of a function is given in terms of the FCT of the function itself. However, the FST of a second derivative is given in terms of the sine transform of the function. There is an additional boundary term $\omega\sqrt{\frac{2}{\pi}}f(0)$.

- **Transformation of partial derivatives:**

  $(i)$ Let $u = u(x,t)$ be a function defined for $x \geq 0$ and $t \geq 0$. If $u(x,t) \to 0$ as $x \to \infty$, and $\mathcal{F}_s[u](\omega,t) = \hat{u}_s(\omega,t)$, then

$$\mathcal{F}_s[u_x](\omega,t) = -\omega\mathcal{F}_c[u](\omega,t).$$

$$\mathcal{F}_c[u_x](\omega,t) = \omega\mathcal{F}_s[u](\omega,t) - \sqrt{\frac{2}{\pi}}u(0,t).$$

  If, in addition, $u_x(x,t) \to 0$ as $x \to \infty$, then

$$\mathcal{F}_s[u_{xx}](\omega,t) = -\omega^2\mathcal{F}_s[u](\omega,t) + \sqrt{\frac{2}{\pi}}\omega u(0,t).$$

$$\mathcal{F}_c[u_{xx}](\omega,t) = -\omega^2\mathcal{F}_c[u](\omega,t) - \sqrt{\frac{2}{\pi}}u_x(0,t).$$

  $(ii)$ If we transform the partial derivative $u_t(x,t)$ (and if the variable of integration in the transformation is $x$), then the transformation is given by

$$\mathcal{F}_s[u_t](\omega,t) = \frac{d}{dt}\{\mathcal{F}_s[u]\}(\omega,t).$$

$$\mathcal{F}_c[u_t](\omega,t) = \frac{d}{dt}\{\mathcal{F}_c[u]\}(\omega,t).$$

  Thus, time differentiation commutes with both the Fourier cosine and sine transformations.

## Practice Problems

1. Find the FST and FCT of the function

$$f(x) = \begin{cases} 1, & 0 \leq x \leq 2, \\ 0, & x > 2. \end{cases}$$

2. If $u = u(x,t)$ and $u(x,t) \to 0$ as $x \to \infty$, then

   (A) $\mathcal{F}_s u_x(\omega,t) = -\omega\mathcal{F}_c[u](\omega,t)$

   (B) $\mathcal{F}_c u_x(\omega,t) = -\frac{2}{\pi}u(0,t) + \omega\mathcal{F}_s[u](\omega,t)$

3. If $u(x,t)$ and $u_x(x,t) \to 0$ as $x \to \infty$, then

   (A) $\mathcal{F}_s[u_{xx}](\omega,t) = -\omega^2\mathcal{F}_s[u](\omega,t) + \sqrt{\frac{2}{\pi}}\omega u(0,t)$

   (B) $\mathcal{F}_c[u_{xx}](\omega,t) = -\omega^2\mathcal{F}_c[u](\omega,t) - \sqrt{\frac{2}{\pi}}u_x(0,t)$