1 Asymptotic Notation

1. Prove that $3n + 2$ is $O(n)$.

   By the definition of big Oh, we need to find constants $c > 0$ and $n_0 \geq 1$ such that

   $$3n + 2 \leq cn, \quad \forall n \geq n_0 \tag{1}$$

   First, we simplify the inequality by moving the term $3n$ to the right hand side to get

   $$2 \leq (c - 3)n, \quad \forall n \geq n_0$$

   Since the right hand side of the inequality needs to be positive, we choose, for example, $c = 4$:

   $$2 \leq (4 - 3)n = n, \quad \forall n \geq n_0$$

   Note that $n \geq 2$ for all values $n \geq 2$, so we choose $n_0 = 2$.

   As we have found constant values $c = 4$ and $n_0 = 2$ that make inequality (1) true, then we have proven that $3n + 2$ is $O(n)$.

2. Prove that $3n + 2$ is $O(n^2)$.

   By the definition of big Oh, we need to find constants $c > 0$ and $n_0 \geq 1$ such that

   $$3n + 2 \leq cn^2, \quad \forall n \geq n_0 \tag{2}$$

   First, we simplify the inequality by moving the term $3n$ to the right hand side to get

   $$2 \leq cn^2 - 3n = (cn - 3)n, \quad \forall n \geq n_0$$

   Since the right hand side of the inequality needs to be positive, we choose, for example, $c = 3$:

   $$2 \leq (3n - 3)n, \quad \forall n \geq n_0$$

   Note that $3n - 3 > 0$ for all $n \geq 2$, and so $(3n - 3)n \geq 6 \geq 2$ for all values $n \geq 2$. Therefore we choose $n_0 = 2$.

   As we have found content values $c = 3$ and $n_0 = 2$ that make inequality (2) true, then we have proven that $3n + 2$ is $O(n)$.

   Note that $3n + 1$ is $O(n)$ and it is also $O(n^2)$. When we determine the order of a function $f(n)$ we try to find the smallest function $g(n)$ such that $f(n)$ is $O(g(n))$.

3. Prove that n^2 is not $O(n)$.

   By the definition of big Oh, if we wanted to prove that n^2 is $O(n)$ we would have to find constants $c > 0$ and $n_0 \geq 1$ such that

   $$n^2 \leq cn, \quad \forall n \geq n_0$$

   Since we need to prove the opposite of the above claim, namely that n^2 is not $O(n)$, we need to show that **there are no constants** $c > 0$ and $n_0 \geq 1$ such that

$$n^2 \leq cn, \quad \forall n \geq n_0$$

Or, equivalently, we need to show that **for all** constants $c > 0$ and $n_0 \geq 1$

$$n^2 > cn, \quad \text{for at least one value } n \geq n_0 \tag{3}$$

This proof is different from the two above ones, in that now we cannot fix the values of c and n_0 as inequality (3) must hold **for all** constants $c > 0$ and $n_0 \geq 1$. What we need to do is to show that for any c and n_0 there is at least one value n that satisfies (3).

Since n must be larger than or equal to n_0 and $n_0 \geq 1$, then n is positive. Hence, we can divide both sides of (3) by n to get

$$n > c, \quad \text{for at least one value } n \geq n_0$$

If we choose, for example, $n = \max\{c, n_0\} + 1$, this value is larger than c and larger than or equal to n_0, so this value of n makes inequality (3) true, and hence n^2 is not $O(n)$.

We also give a different proof that n^2 is not $O(n)$ which uses contradiction: Assume that n^2 is O(n) and derive a contradiction from that assumption. The proof is as follows. First, assume that n^2 is $O(n)$. This means that there are constants $c > 0$ and $n_0 \geq 1$ such that

$$n^2 \leq cn, \quad \forall n \geq n_0 \tag{4}$$

Since $n \geq n_0 \geq 1$, then n is positive so we can divide both sides of (4) by n to get

$$n \leq c, \quad \forall n \geq n_0 \tag{5}$$

Note that regardless of the value of c, n cannot be less than or equal to c for all $n \geq n_0$ because n is a function that grows without bound. Hence, for example if $n = \max\{c, n_0\} + 1$, this value is larger that or equal to n_0, **but** it is also larger than c, hence inequality (5) is not true and we have derived a contradiction!