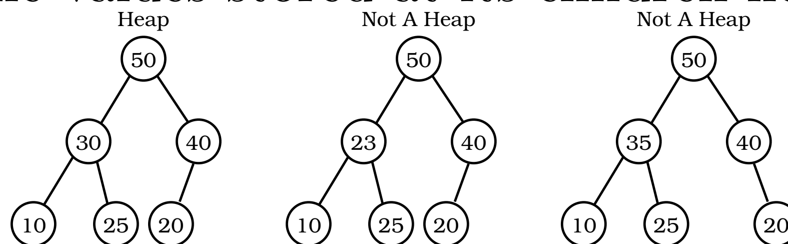
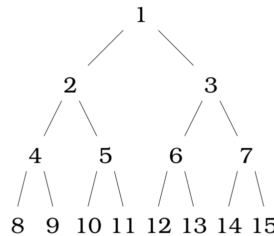
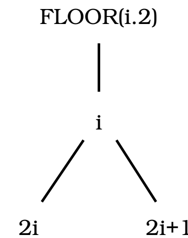
# Max Heaps

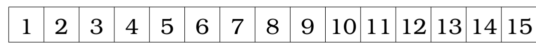
- Each node stores one value, but the values may be repeated (i.e., it is a multiset, rather than a set).
- Complete binary tree (All possible nodes at each level are present except possibly for the last level. All the missing nodes in the last level, if any, are located to the right of the nodes that are present.)
- The value stored at node $v$ is at least as large as the values stored at its children nodes.



# Representation Without Pointers




```
template<class T>
class MaxHeap {
   public:
      MaxHeap(int MaxHeapSize = 10);
      ~MaxHeap() {delete [] heap;}
   private:
      int CurrentSize, MaxSize;
      T *heap;  // element array };
template<class T>
MaxHeap<T>::MaxHeap(int MaxHeapSize)
{ MaxSize = MaxHeapSize;
   heap = new T[MaxSize+1];
   CurrentSize = 0; }
```

## Height of Heap with $n$ nodes

- Full Binary Tree (no missing nodes).
  - Number of nodes at level 1 is $2^0$, at level 2 is $2^1$, at level 3 is $2^2$, ..., at level $h$ is $2^{h-1}$.
  - Therefore, $n = \sum_{i=0}^{h-1} 2^i$
  - So, $n = 2^h - 1$, and $n + 1 = 2^h$.
  - So, $h = log_2(n + 1)$.
  - Or $h$ is $O(log\ n)$.
- Not Full Binary Tree (missing nodes).
  - Fill it up. Number of nodes is $m < 2n$.
  - Therefore, $h = log_2(m + 1) < log_2(2n + 1)$.
  - Or $h$ is $O(log\ n)$.

## **Insert** $x$

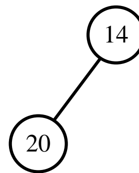
- Assign $x$ to the next available position.
- If $x$ is greater than the value of its parent then swap them.
- repeat the above operation till you reach the root or it does not hold.

# Example for Insert

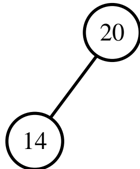
EMPTY HEAP

Insert 14

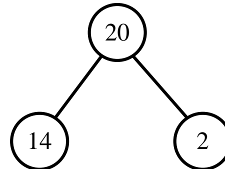
Insert 20

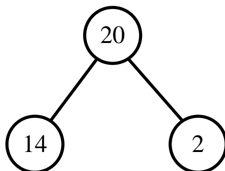



Move Up

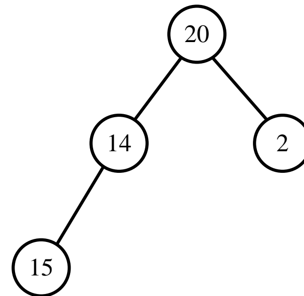



Insert 2

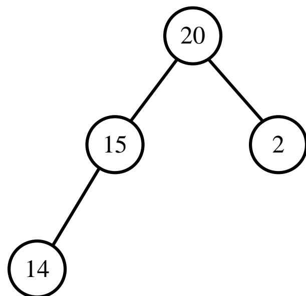


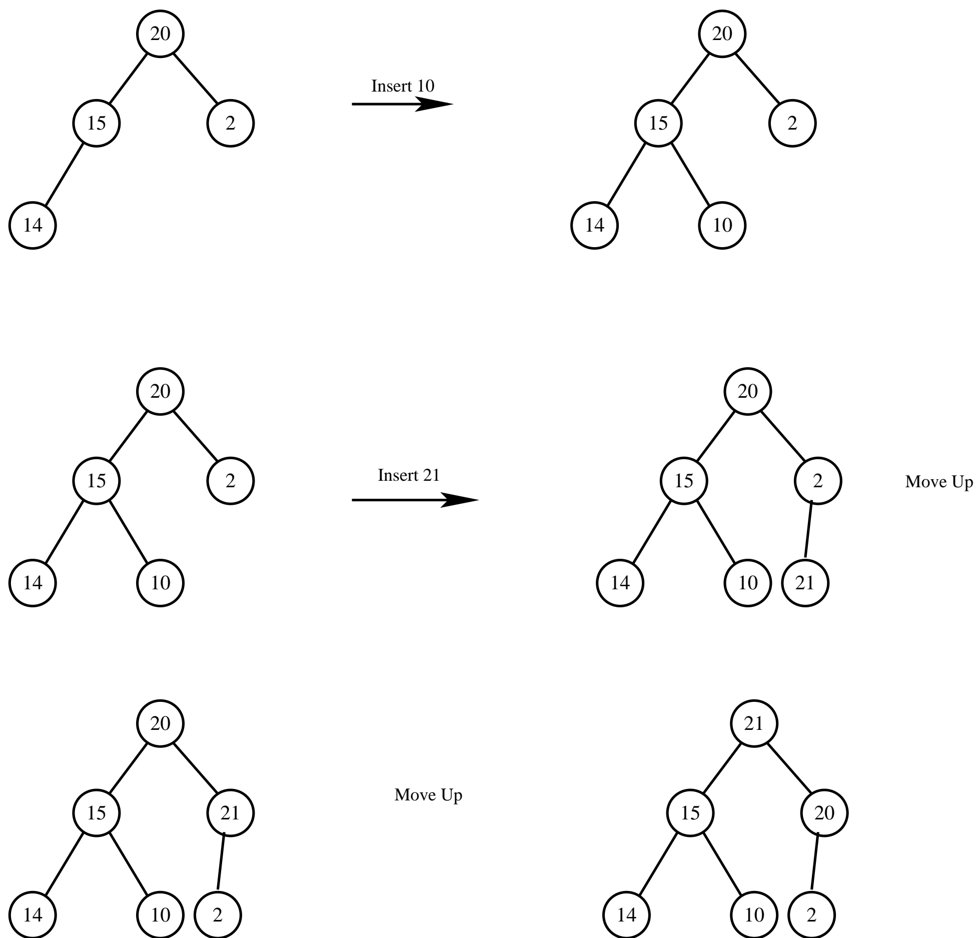

Insert 15



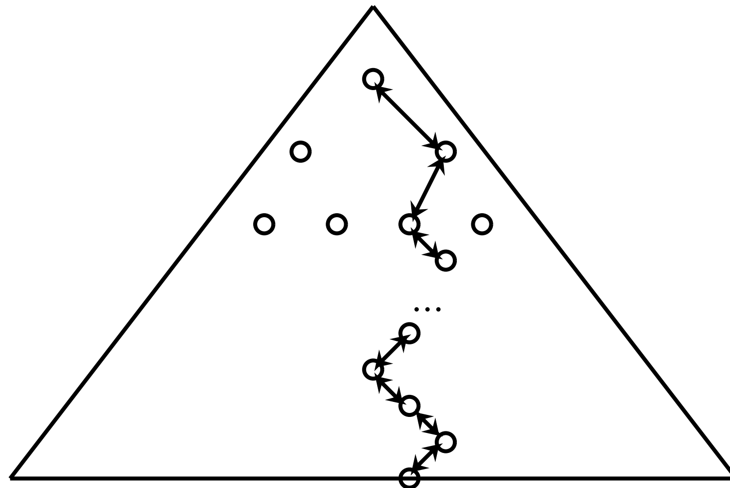

Move Up



# Example for Insert



## General Idea



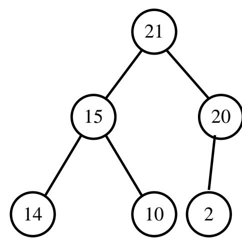

Just move them down (instead of swap) and store $x$ in the appropriate place.

```
template<class T>
MaxHeap<T>& MaxHeap<T>::Insert(const T& x)
{// Insert x into the max heap.
   if (CurrentSize == MaxSize)
      throw NoMem(); // no space

   // find place for x
   // i starts at new leaf and moves up tree
   int i = ++CurrentSize;
   while (i != 1 && x > heap[i/2]) {
      // cannot put x in heap[i]
      heap[i] = heap[i/2]; // move down
      i /= 2;} // move to parent

   heap[i] = x;
}
```

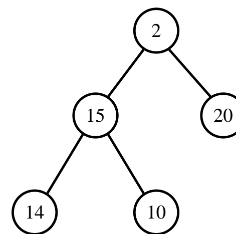
Time Complexity is $O(log\ n)$.

# Deletion

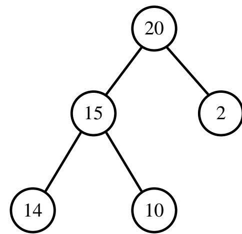
- Copy the value in the root and that is what will be returned.
- Move the last element to the root.
- If one of its children has a larger value, then move it to the child with largest value.
- Repeat the abve until you reach a leaf or the above condition does not hold.



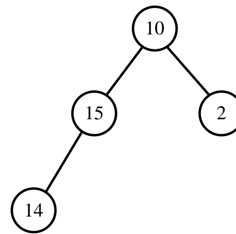

DeleteMax



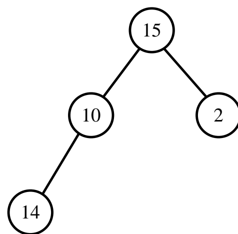

Move Down



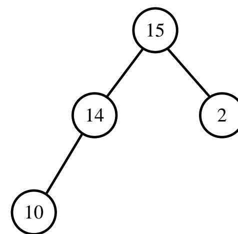

DeleteMax

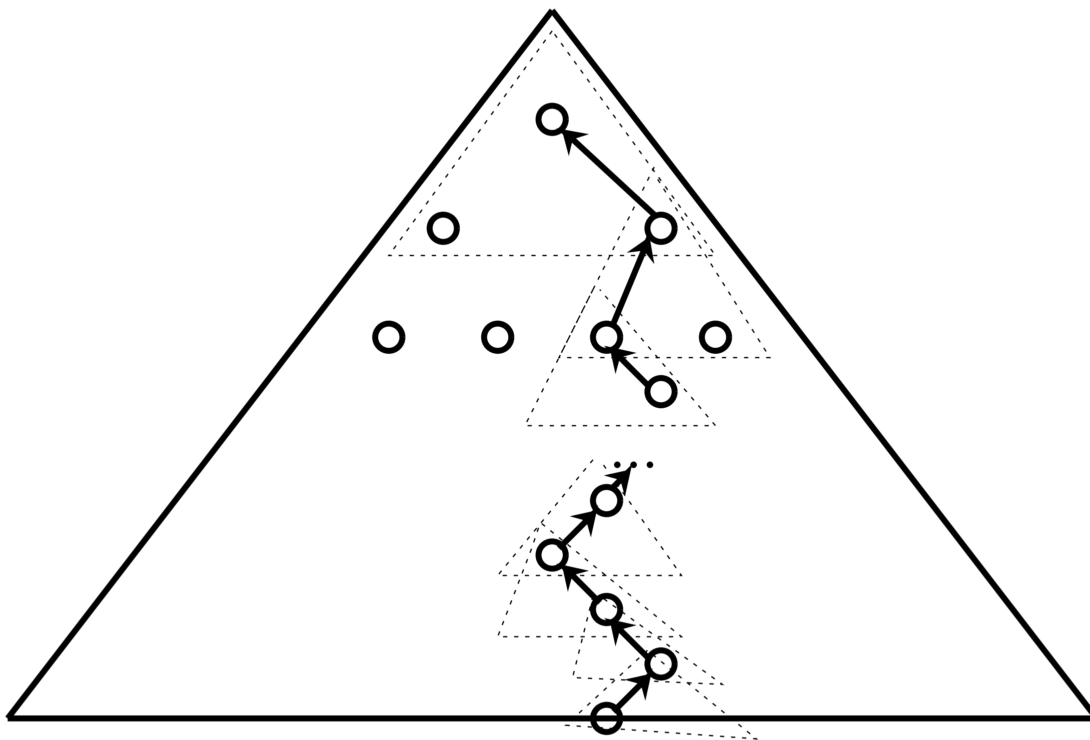



Move Down




Move Down



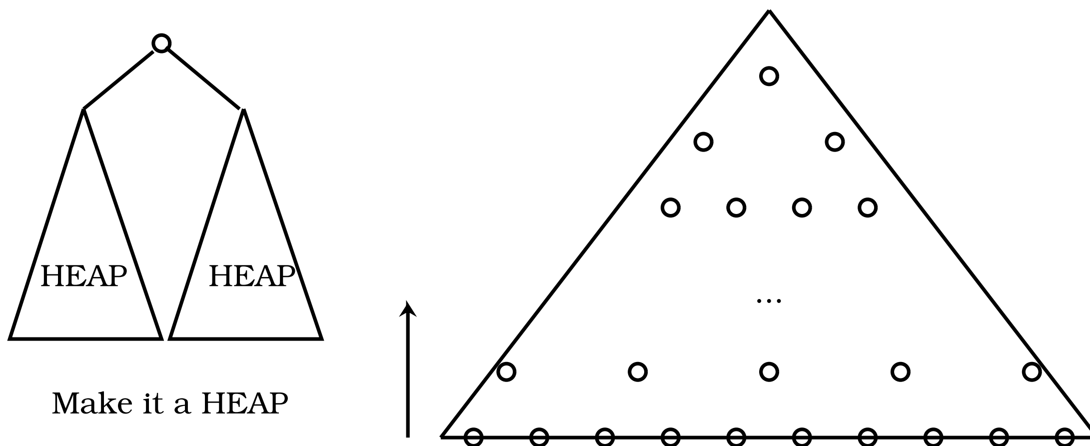


```
template<class T>
MaxHeap<T>& MaxHeap<T>::DeleteMax(T& x)
{ if (CurrentSize == 0)
      throw OutOfBounds(); // empty
   x = heap[1]; // max element
   T y = heap[CurrentSize--]; // last element
   int i = 1,  // current node of heap
       ci = 2; // child of i
   while (ci <= CurrentSize) {
      if (ci < CurrentSize &&
          heap[ci] < heap[ci+1]) ci++;
      // can we put y in heap[ci]?
      if (y >= heap[ci]) break;   // yes
      // no
      heap[i] = heap[ci]; // move child up
      i = ci;  // move down a level
      ci *= 2; }
   heap[i] = y;
 }
```

Time Complexity is $O(log\ n)$.

# Heap Initialization (vs $n$ Sequential Inserts

- By inserting one at a time takes $O(n\ log\ n)$ time (actually $\Omega$ too).
- New procedure (making the heap bottom-up) takes $O(n)$ Time.




Time Complexity

At level $j$ there are $2^{j-1}$ vertices and making that subtree a heap takes $h - j$ operations (assume complete tree).

$$T(n) = \sum_{j=1}^{h} 2^{j-1}(h-j)$$

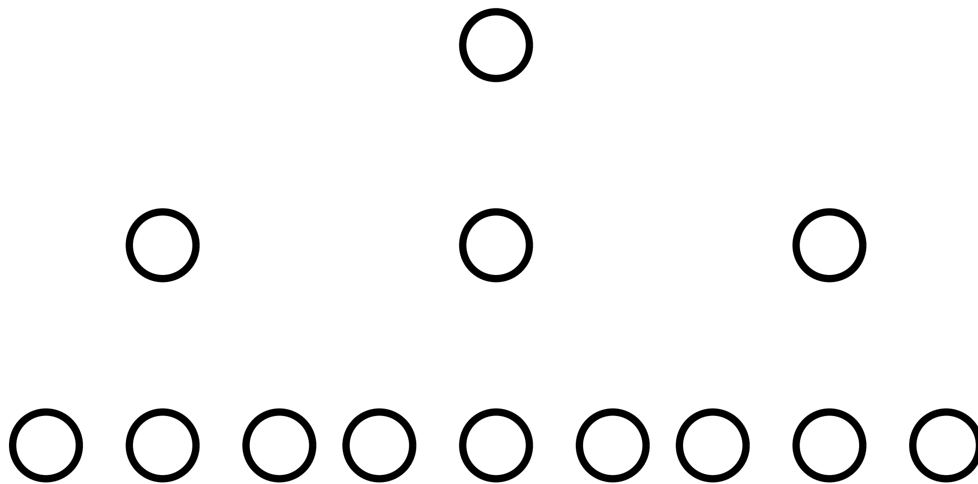
$$
\begin{aligned}
T(n) &= \sum_{j=1}^{h} 2^{j-1}(h-j) \\
&= \sum_{i=0}^{h-1} 2^{h-1-i}(i) \\
&= 2^{h-1} \sum_{i=0}^{h-1} \frac{i}{2^i} \\
&= 2^{h-1} \cdot \frac{2^h - 1 - h}{2^{h-1}} \\
&= 2^h - h - 1 \\
&= O(n)
\end{aligned}
$$

```
template<class T>
void MaxHeap<T>::Initialize(T a[], int size,
                                int ArraySize)
{  delete [] heap;
   heap = a;
   CurrentSize = size;
   MaxSize = ArraySize;
   for (int i = CurrentSize/2; i >= 1; i--) {
      T y = heap[i]; // root of subtree
      int ci = 2*i; // parent of c is target
                    // location for y
      while (ci <= CurrentSize) {
         // heap[ci] should be larger sibling
         if (ci < CurrentSize &&
             heap[ci] < heap[ci+1]) ci++;
         // can we put y in heap[ci]?
         if (y >= heap[ci]) break;  // yes
         // no
         heap[ci/2] = heap[ci]; // move child up
         ci *= 2; // move down a level }
      heap[ci/2] = y;} }
```

## Other Operations

- Decrease Value $O(\log n))$.
- Increase Value $O(\log n))$.
- Delete element (if you know its position) $O(\log n))$.
- Delete element (if you do NOT know its position) $O(n)$.

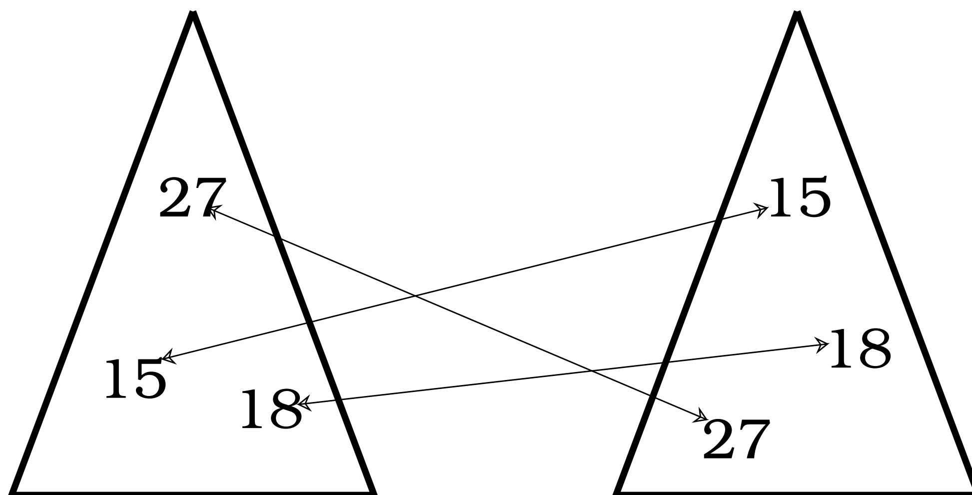
# Ternary Heaps (d-heaps with d=3)



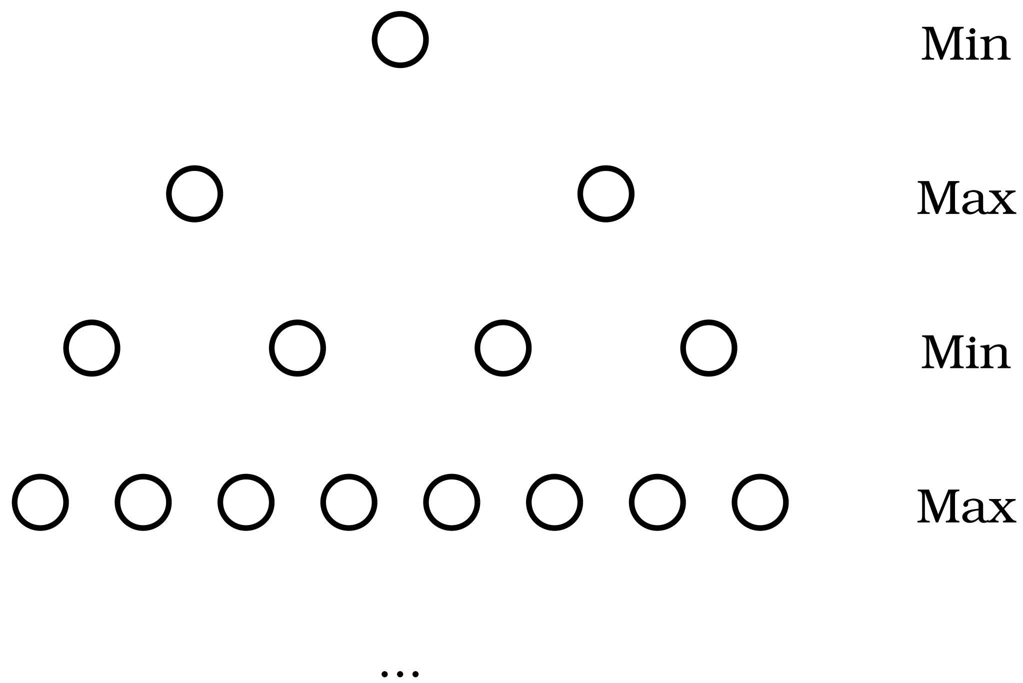
...

Height is $log_3\ n$

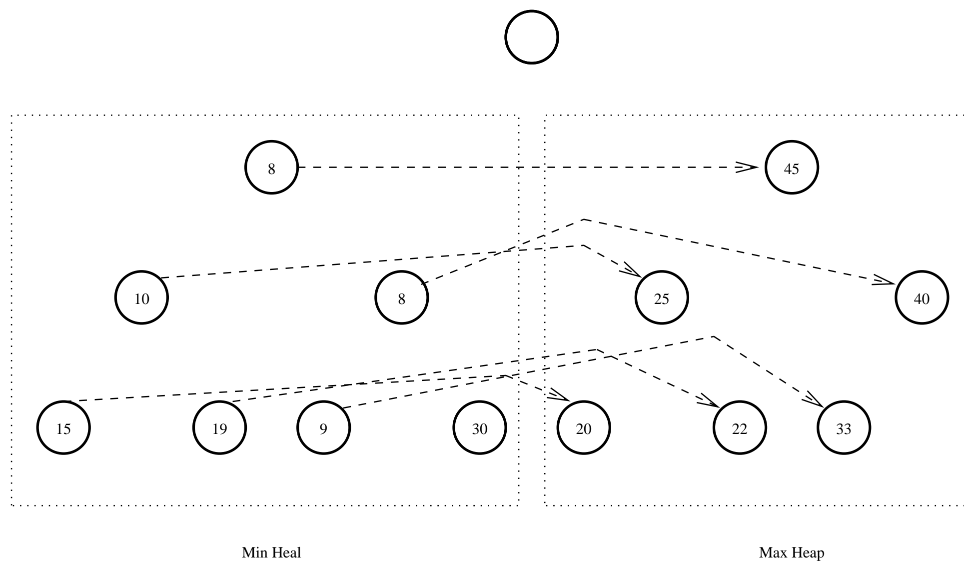
# Matching Heaps



# Min-Max Heaps



# Deaps




Dotted arrow $a \to b$ means that $a \leq b$.