The Binomial Theorem[a]

Theorem 16

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}.$$

- $(x+y)^n = \overbrace{(x+y)(x+y)\cdots(x+y)}^{n}$.
- Each term must have the form $x^i y^{n-i}$.
- There are $\binom{n}{i}$ ways to pick i x's and $n-i$ y's.

[a] Attributed to Newton. First appeared in a book by Colin Maclaurin (1698–1746).

Corollaries of the Binomial Theorem

$$2^n = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}. \tag{4}$$

- Set $x = y = 1$ in the binomial theorem.

For odd n,

$$\begin{aligned} 2^{n-1} &= \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{\frac{n-1}{2}} \\ &= \binom{n}{\frac{n+1}{2}} + \binom{n}{\frac{n+3}{2}} + \cdots + \binom{n}{n}. \end{aligned} \tag{5}$$

- Because $\binom{n}{r} = \binom{n}{n-r}$.

Corollaries of the Binomial Theorem (continued)

- Set $x = 1$ and $y = -1$ in the binomial theorem to obtain

$$\binom{n}{0} - \binom{n}{1} + \cdots + (-1)^n \binom{n}{n} = 0.$$

- As a by-product, when $n > 0$,

$$\begin{aligned} & \binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots \\ = & \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots \\ = & 2^{n-1}. \end{aligned} \tag{6}$$

Corollaries of the Binomial Theorem (continued)

$$\sum_{i=n+1}^{2n+1} \binom{2n+1}{i} = 2^{2n} \tag{7}$$

because

$$\begin{aligned} 2^{2n+1} &= \sum_{i=0}^{2n+1} \binom{2n+1}{i} = \sum_{i=0}^{n} \binom{2n+1}{i} + \sum_{i=n+1}^{2n+1} \binom{2n+1}{i} \\ &= \sum_{i=0}^{n} \binom{2n+1}{2n+1-i} + \sum_{i=n+1}^{2n+1} \binom{2n+1}{i} \\ &= 2 \sum_{i=n+1}^{2n+1} \binom{2n+1}{i}. \end{aligned}$$

Corollaries of the Binomial Theorem (continued)

$$\binom{2n}{n} = \sum_{i=0}^{n} \binom{n}{i}^2. \tag{8}$$

- Consider

$$\begin{aligned} f(x) &= (1+x)^n (1+x^{-1})^n \\ &= \overbrace{(1+x)\cdots(1+x)}^{n}\,\overbrace{(1+x)\cdots(1+x^{-1})}^{n} \\ &= \sum_{i=-n}^{n} f_i x^i. \end{aligned}$$

The Proof (concluded)

- Concentrate on the constant term f_0 of $f(x)$.
- $\binom{n}{i}^2$ is the number of ways to pick i x's and i x^{-1}'s.
- So $f_0 = \sum_{i=0}^{n} \binom{n}{i}^2$.
- Rewrite $f(x)$ as

$$f(x) = (1+x)^n (1+x)^n x^{-n} = x^{-n}(1+x)^{2n}.$$

- The constant term in $f(x)$ is the coefficient of x^n in $(1+x)^{2n}$.
- So $f_0 = \binom{2n}{n}$.

Alternative Proof for Eq. (8) on p. 53

- Consider a $2n$-step binomial random walk that ends at the origin.
- There are $\binom{2n}{n}$ such walks by Eq. (2) on p. 42.
- Consider a walk that reaches position i at step n, where $n + i$ is even.
- There are $\binom{n}{(n+i)/2}^2$ such walks by Eq. (2) on p. 42.
- So

$$\binom{2n}{n} = \sum_{i=-n,-n+2,\ldots,n} \binom{n}{(n+i)/2}^2 = \sum_{i=0}^{n} \binom{n}{i}^2.$$

A Combinatorial Proof[a] for Eq. (8) on p. 53

- There are $\binom{2n}{n}$ ways to pick n objects out of $2n$ distinct objects.
- Now, divide the $2n$ objects into two groups equally.
- There are $\binom{n}{i}\binom{n}{n-i}$ ways to pick i objects from the first group and the remaining $n-i$ objects from the second.
- As i can vary from 0 to i,

$$\binom{2n}{n} = \sum_{i=0}^{n} \binom{n}{i}\binom{n}{n-i} = \sum_{i=0}^{n} \binom{n}{i}\binom{n}{i}.$$

[a]Contributed by Mr. Gong-Ching Lin (`B00703082`) on February 27, 2012.

Corollaries of the Binomial Theorem (continued)

$$\binom{2n}{n} = \sum_{i=0}^{2n} (-1)^{n+i} \binom{2n}{i}^2 .$$

- Consider

$$\begin{aligned} g(x) &= (1+x)^{2n}(1-x^{-1})^{2n} \\ &= \overbrace{(1+x)\cdots(1+x)}^{2n}\overbrace{(1-x^{-1})\cdots(1-x^{-1})}^{2n} \\ &= \sum_{i=-2n}^{2n} g_i x^i . \end{aligned}$$

The Proof (concluded)

- Concentrate on the constant term g_0 of $g(x)$.
- $\binom{2n}{i}^2$ is the number of ways to pick i x's and i $-x^{-1}$'s.
- So $g_0 = \sum_{i=0}^{2n} (-1)^i \binom{2n}{i}^2$.
- Rewrite $g(x)$ as $(x - x^{-1})^{2n} = x^{-2n}(x^2 - 1)^{2n}$.
- The constant term in $g(x)$ is the coefficient of x^{2n} in $(x^2 - 1)^{2n}$.
- So $g_0 = (-1)^n \binom{2n}{n}$.

Corollaries of the Binomial Theorem (concluded)

$$\sum_{i=1}^{n} i \binom{n}{i} = n2^{n-1}.$$

- Differentiate $(1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i$ to obtain

$$n(1+x)^{n-1} = \sum_{i=1}^{n} i \binom{n}{i} x^{i-1}.$$

- Now set $x = 1$.[a]

[a] An alternative proof to avoid calculus is to observe that $i\binom{n}{i} = n\binom{n-1}{i-1}$. So $\sum_{i=1}^{n} i\binom{n}{i} = \sum_{i=1}^{n} n\binom{n-1}{i-1} = n\sum_{i=1}^{n}\binom{n-1}{i-1} = n2^{n-1}$. Contributed by Mr. Gong-Ching Lin (B00703082) on February 27, 2012.

Binary Strings with Even Weight

- Consider a binary string $x_1 x_2 \cdots x_n$.
  - The **weight** of $x_1 x_2 \cdots x_n$ is defined as $\sum_i x_i$.
- There are 2^n strings.
- Among them,

$$\sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n}{2i} \tag{9}$$

  have an even weight.
  - 1 occurs in $2i$ positions.
  - E.g., $\binom{5}{0} + \binom{5}{2} + \binom{5}{4} = 16$, and $\binom{7}{0} + \binom{7}{2} + \binom{7}{4} + \binom{7}{6} = 64$.
- But Eq. (9) equals 2^{n-1} by Eq. (6) on p. 51.

Majority Decision

- In a court with $2n+1$ judges, in how many ways can a majority "yes" decision be handed down?
  - Any vote has a majority; hence we consider cases when the majority vote "yes."
- There are $\binom{2n+1}{i}$ ways such that i judges vote "yes."
- From Eq. (7) on p. 52, the desired answer is

$$\sum_{i=n+1}^{2n+1} \binom{2n+1}{i} = 2^{2n}$$

Ways To Merge Sets

What is the number of ways to merge members of $\{\{1\},\{2\},\ldots,\{n\}\}$ to form $\{\{1,2,\ldots,n\}\}$ in $n-1$ steps?

- Each merge involves two members.
- For example, the number is 3 when $n=3$:

$$
\begin{aligned}
\{\{1\},\{2\},\{3\}\} &\rightarrow \{\{1,2\},\{3\}\} \rightarrow \{\{1,2,3\}\},\\
\{\{1\},\{2\},\{3\}\} &\rightarrow \{\{1,3\},\{2\}\} \rightarrow \{\{1,2,3\}\},\\
\{\{1\},\{2\},\{3\}\} &\rightarrow \{\{2,3\},\{1\}\} \rightarrow \{\{1,2,3\}\}.
\end{aligned}
$$

Ways To Merge Sets (concluded)

- The ith step begins with $n - i + 1$ members.
- There are $\binom{n-i+1}{2}$ ways to pick the two members.
- The desired number is thus

$$\prod_{i=1}^{n-1} \binom{n-i+1}{2} = \frac{n!\,(n-1)!\cdots 2!}{2^{n-1}(n-2)!\,(n-3)!\cdots 1!} = \frac{n!\,(n-1)!}{2^{n-1}}.$$

The Multinomial Theorem

Theorem 17

$$(x_1 + x_2 + \cdots + x_t)^n$$

$$= \sum_{0 \leq n_1, n_2, \ldots, n_t \leq n, \sum_i n_i = n} \frac{n!}{n_1!\, n_2! \cdots n_t!}\, x_1^{n_1} x_2^{n_2} \cdots x_t^{n_t}.$$

- Expand $(x_1 + x_2 + \cdots + x_t)^n$.
- Each term in the expansion must have the form

$$(\text{coefficient}) \times x_1^{n_1} x_2^{n_2} \cdots x_t^{n_t},$$

where $0 \leq n_1, n_2, \ldots, n_t \leq n$ and $\sum_i n_i = n$.

The Proof (concluded)

- The coefficient of $x_1^{n_1} x_2^{n_2} \cdots x_t^{n_t}$ equals the number of ways to pick n_1 x_1's, n_2 x_2's, and so on.
- By Eq. (1) on p. 19, there are

$$\binom{n}{n_1, n_2, \ldots, n_t} \equiv \frac{n!}{n_1!\, n_2! \cdots n_t!}$$

ways.

Coefficient of $a^2b^3c^2d^5$ in $(a + 2b - 3c + 2d + 5)^{16}$

- Make $x_1 = a$, $x_2 = 2b$, $x_3 = -3c$, $x_4 = 2d$, and $x_5 = 5$ symbolically.
- The coefficient of $a^2(2b)^3(-3c)^2(2d)^5 5^4$ is

$$\binom{16}{2,3,2,5,4} = \frac{16!}{2!\,3!\,2!\,5!\,4!} = 302,702,400$$

by the multinomial theorem.

- The desired coefficient is then

$$\begin{aligned} & 302,702,400 \times 2^3 \times (-3)^2 \times 2^5 \times 5^4 \\ = \;& 435,891,456,000,000. \end{aligned}$$

Distinct Objects into Identical Containers

Corollary 18 *There are* $\frac{(rn)!}{(r!)^n n!}$ *ways to distribute* rn *distinct objects into* n *identical containers so that each container contains exactly* r *objects.*

- Consider $(x_1 + x_2 + \cdots + x_n)^{rn}$.
  - Each object is associated with one $x_1 + x_2 + \cdots + x_n$, and each x_i denotes its choice of containers.
  - Each object has a choice of being assigned to one of the n containers.
- What does the coefficient of $x_1^r x_2^r \cdots x_n^r$ mean?

Distinct Objects into Identical Containers (concluded)

- It is the number of ways rn distinct objects can be distributed into n *distinct* containers, each of which contains r objects.
- By Theorem 17 (p. 64), it is

$$\binom{rn}{r, r, \ldots, r} \equiv \frac{(rn)!}{r!\, r! \cdots r!}.$$

- Finally, divide the above count by $n!$ to remove the identities of the containers.

Corollary 19 $\frac{(rn)!}{(r!)^n n!}$ *is an integer.*

- Immediate from Corollary 18 (p. 67).

An Alternative Proof of Corollary 19 (p. 68)[a]

$$
\begin{aligned}
& \frac{(rn)!}{(r!)^n n!} \\
= \; & \frac{1}{n!} \frac{(rn)!}{[\,r(n-1)\,]!\, r!} \frac{[\,r(n-1)\,]!}{[\,r(n-2)\,]!\, r!} \cdots \frac{[\,r(1)\,]!}{[\,r(n-n)\,]!\, r!} \\
= \; & \frac{\prod_{k=0}^{n-1} \binom{r(n-k)}{r}}{n!} \\
= \; & \prod_{k=0}^{n-1} \frac{\binom{r(n-k)}{r}}{n-k} \\
= \; & \prod_{k=0}^{n-1} \binom{r(n-k)-1}{r-1}.
\end{aligned}
$$

[a] Contributed by Mr. Ansel Lin (B93902003) on September 20, 2004.

Combinations with Repetition

Theorem 20 *Suppose there* n distinct *objects and* r *is an integer,* $0 \leq r$*. The number of selections of* r *of these objects,* with repetition, *is*

$$C(n+r-1, r) = \binom{n+r-1}{r}.$$

- Note that the order of selection is not important.
- Permute

$$\overbrace{xx \cdots x}^{r} \overbrace{| \, | \cdots |}^{n-1}.$$

- Think of the ith interval as containing the ith type of objects.

The Proof (concluded)

- So

$$xx \,|\, xxx \,|\, x \,|\,|\,|\,|$$

  means, out of 7 distinct objects, we pick 2 type-1 objects, 3 type-2 objects, and 1 type-3 object.

- Our goal equals the number of permutations of

$$\overbrace{xx \cdots x}^{r}\overbrace{|\,|\cdots|}^{n-1}.$$

Integer Solutions of a Linear Equation

The following three problems are equivalent (they all equal $\binom{n+r-1}{r}$; see p. 432 and p. 434 for alternative proofs):

1. The number of selections, with repetition, of size r from a collection of n distinct objects (Theorem 20 on p. 70).
2. The number of ways r *identical* objects can be distributed among n *distinct* containers.[a]
3. The number of nonnegative *integer* solutions of $x_1 + x_2 + \cdots + x_n = r$.

---

[a] The case of distinct objects and identical containers will be covered on p. 227 (see p. 67 for a special case).

Application: The Multinomial Theorem

- The number of distinct terms in $(x_1 + x_2 + \cdots + x_t)^n$ is equal to the number of nonnegative integer solutions to $n_1 + n_2 + \cdots + n_t = n$.
  - Each term has the form $x_1^{n_1} x_2^{n_2} \cdots x_t^{n_t}$ such that
    * $n_1 + n_2 + \cdots + n_t = n$, and
    * $0 \leq n_1, n_2, \ldots, n_t$.
- The number of terms is therefore

$$\binom{n+t-1}{n}.$$

from the equivalencies on p. 72.

Positive Integer Solutions of a Linear Equation

- Consider $x_1 + x_2 + \cdots + x_n = r$, where $x_i > 0$ for $1 \leq i \leq n$.
- It is equivalent to $x'_1 + x'_2 + \cdots + x'_n = r - n$, where $x'_i \geq 0$ for $1 \leq i \leq n$
  - By making $x'_i = x_i - 1$.
- The number of solutions to the original problem is therefore (p. 72)

$$\binom{n + (r - n) - 1}{r - n} = \binom{r - 1}{r - n}. \tag{10}$$

Application: Subsets with Restrictions

How many n-element subsets of $\{1, 2, \ldots, r\}$ contain no consecutive integers?

- Say $r = 4$ and $n = 2$.
- Then the valid 2-element subsets of $\{1, 2, 3, 4\}$ are

$$\{1, 3\}, \{1, 4\}, \{2, 4\}.$$

The Proof (continued)

- Let $i_0 = 1$ and $i_{n+1} = r$ as placeholders.
- For each valid subset $\{i_1, i_2, \ldots, i_n\}$, where $1 \leq i_1 < i_2 < \cdots < i_n \leq r$, define $d_k = i_{k+1} - i_k$.
- Then, by telescoping,

$$d_0 + d_1 + \cdots + d_n = i_{n+1} - i_0 = r - 1.$$

- Observe that

$$0 \leq d_0, d_n$$
$$2 \leq d_1, d_2, \ldots, d_{n-1}.$$

The Proof (concluded)

- So equivalently,

$$d_0' + d_1' + \cdots + d_n' = r - 1 - 2(n-1)$$

  with $0 \leq d_0', d_1', \ldots, d_n'$.

- The answer to the desired number is

$$\binom{(n+1) + (r - 1 - 2(n-1)) - 1}{r - 1 - 2(n-1)} = \binom{r - n + 1}{r - 2n + 1}$$

  from the equivalencies on p. 72.

Application: Political Majority

In how many ways can $2n+1$ seats in a parliament be divided among 3 parties so that the coalition of any 2 parties form a majority?

- This is a problem of distributing identical objects (the seats) among distinct containers (the parties) (p. 72).
- So without the majority condition, the number is $\binom{3+(2n+1)-1}{2n+1} = \binom{2n+3}{2}$.
- Observe that the majority condition is violated if and only if a party gets $n+1$ or more seats.

The Proof (concluded)

- If a given party gets $n+1$ or more seats, the number of ways of distributing the seats is $\binom{3+n-1}{n} = \binom{n+2}{2}$.
  - Allocate $n+1$ seats to that party before allocating the remaining n seats to the 3 parties.
  - Refer to p. 72 for the formula.
- The desired number of no dominating party is

$$\binom{2n+3}{2} - 3\binom{n+2}{2} = \frac{n}{2}(n+1) = \binom{n+1}{2}. \quad (11)$$

Integer Solutions of a Linear Inequality

- Consider $x_1 + x_2 + \cdots + x_n \leq r$, where $x_i \geq 0$ for $1 \leq i \leq n$.
- It is equivalent to $x_1 + x_2 + \cdots + x_n + x_{n+1} = r$, where $x_i \geq 0$ for $1 \leq i \leq n+1$
- The number of *integer* solutions of the original inequality is therefore (p. 72)
$$\binom{n+r}{r}. \tag{12}$$

Integer Solutions of a Strict Linear Inequality

- Consider $x_1 + x_2 + \cdots + x_n < r$, where $x_i \geq 0$ for $1 \leq i \leq n$.
- It is equivalent to $x_1 + x_2 + \cdots + x_n \leq r - 1$, where $x_i \geq 0$ for $1 \leq i \leq n$.
- By Eq. (12) on p. 80, the number of nonnegative integer solutions is

$$\binom{n+r-1}{r-1}.$$

Compositions of Positive Integers

- Let m be a positive integer.
- A **composition** for m is a sum of positive integers whose order is relevant and which sum to m.
- For $m = 3$, the number of compositions is four: 3, $2 + 1$, $1 + 2$, $1 + 1 + 1$.
- For $m = 4$, the number of compositions is eight: 4, $3 + 1$, $2 + 2$, $1 + 3$, $1 + 1 + 2$, $1 + 2 + 1$, $2 + 1 + 1$, $1 + 1 + 1 + 1$.
- Is the number of compositions for general m equal to 2^{m-1}?

The Number of Compositions

Theorem 21 *The number of compositions for* $m > 0$ *is* 2^{m-1}.

- Every composition with i summands corresponds to a positive integer solution to $x_1 + x_2 + \cdots + x_i = m$.
- Hence the solution number is $\binom{m-1}{m-i}$ by Eq. (10) on p. 74.
- The total number of compositions is $\sum_{i=1}^{m} \binom{m-1}{m-i} = 2^{m-1}$ by Eq. (4) on p. 50.

Palindromes of Positive Integers

- Let m be a positive integer.
- A palindrome for m is a composition for m that reads the same left to right as right to left.
  - For $m = 4$, the number of palindromes is four: $\boxed{4}$, $1 + \boxed{2} + 1$, $2 \boxed{+} 2$, $1 + 1 \boxed{+} 1 + 1$.
  - For $m = 5$, the number of palindromes is four: $\boxed{5}$, $1 + \boxed{3} + 1$, $2 + \boxed{1} + 2$, $1 + 1 + \boxed{1} + 1 + 1$.
  - The center elements are boxed above.
- The numbers to the left of the center element mirror those to the right.

The Number of Palindromes

Theorem 22 *The number of palindromes for* $m > 0$ *is* $2^{\lfloor m/2 \rfloor}$.

- Assume m is even first.
- The central element of a composition of m can be $m, m-2, \ldots, 2$ or "+" (we will think of it as 0).
- When the central element is m, the number of palindromes is clearly 1.
- When the central element is the even number $0 \leq i < m$, the numbers to its left sum to $(m-i)/2$.

The Proof (concluded)

- Hence the number of palindromes is $2^{(m-i)/2-1}$ by Theorem 21 (p. 83).
- The total number of palindromes for m is thus

$$1 + 1 + 2 + 2^2 + \cdots + 2^{(m-2)/2-1} + 2^{m/2-1} = 2^{m/2}.$$

- Follow the same argument when m is odd to obtain a count of $2^{(m-1)/2}$.

Runs

- Consider a permutation of 10 Os and 5 Es:

$$\texttt{O O E O O O O E E E O O O E O}.$$

- It has 7 **runs**:

$$\underbrace{\texttt{O O}}_{\text{run}}\ \underbrace{\texttt{E}}_{\text{run}}\ \underbrace{\texttt{O O O O}}_{\text{run}}\ \underbrace{\texttt{E E E}}_{\text{run}}\ \underbrace{\texttt{O O O}}_{\text{run}}\ \underbrace{\texttt{E}}_{\text{run}}\ \underbrace{\texttt{O}}_{\text{run}}.$$

- In general, a run is a maximal consecutive list of identical objects.

The Number of Runs

Theorem 23 *There are*

$$\binom{m-1}{m-\lceil r/2\rceil}\binom{n-1}{n-\lfloor r/2\rfloor}+\binom{n-1}{n-\lceil r/2\rceil}\binom{m-1}{m-\lfloor r/2\rfloor}$$

ways that m identical objects of type 1 and n identical objects of type 2 can give rise to r runs.

- Suppose the run starts with a type-1 object.
- Let x_i count the number of type-1 objects in the ith run, $i = 1, 3, \ldots, 2\lceil r/2\rceil - 1$.
- The number of runs with the said counts $x_1, x_2, \ldots$ equals the number of positive-integer solutions to $x_1 + x_3 + \cdots + x_{2\lceil r/2\rceil-1} = m$.

The Proof (continued)

- By Eq. (10) on p. 74, the number of solutions equals

$$\binom{m-1}{m-\lceil r/2 \rceil}.$$

- Let x_i count the number of type-2 objects in the ith run, $i = 2, 4, \ldots, 2\lfloor r/2 \rfloor$.
- The number of runs with the said counts $x_1, x_2, \ldots$ equals that of positive-integer solutions to $x_2 + x_4 + \cdots + x_{2\lfloor r/2 \rfloor} = n$:

$$\binom{n-1}{n-\lfloor r/2 \rfloor}.$$

The Proof (concluded)

- Therefore the number of runs that start with a type-1 object equals

$$\binom{m-1}{m-\lceil r/2\rceil}\binom{n-1}{n-\lfloor r/2\rfloor}.$$

- Repeat the argument for the case where the 1st run starts with a type-2 object.

- The count is

$$\binom{n-1}{n-\lceil r/2\rceil}\binom{m-1}{m-\lfloor r/2\rfloor}$$

(by swapping m and n).

The Catalan[a] Numbers (1838)

- A binomial random walk starts at the origin.
- What is the number of ways it can end at the origin in $2n$ steps *without* being in the negative territory?
- It is equivalent to the number of ways $\overbrace{RR\cdots R}^{n}\overbrace{LL\cdots L}^{n}$ can be permuted so that no prefix has more Ls than Rs.
  - A left move lowers the position, whereas a right move increases the position.

[a]Eugène Charles Catalan (1814–1894).

The Catalan Numbers (concluded)

- For example,

$$\overbrace{\overbrace{\overbrace{\overbrace{\overbrace{\overbrace{\overbrace{\overbrace{R}^{1}\, L}^{0} R}^{1} L}^{0} R}^{1} R}^{2} L}^{1} L}^{0}.$$

Formula for the Catalan Numbers

The number is

$$b_n = \binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{n+1}\binom{2n}{n}, n \geq 1.$$

with $b_0 = 1$.

- $\overbrace{RR\cdots R}^{n}\overbrace{LL\cdots L}^{n}$ can be permuted in $\binom{2n}{n}$ ways.
- Some of the permutations are illegal, such as $RLLLRR$.
- We now prove that $\binom{2n}{n-1}$ of the permutations are illegal.

The Proof (concluded)

- For every illegal permutation, we consider the first L move that makes the particle land at -1.
  - Such as $RL\boxed{\text{L}}LRR$.
- Swap L and R for this offending L and all earlier moves.
  - Such as $\boxed{\text{L}}\boxed{\text{R}}\boxed{\text{R}}LRR$.
- The result is some permutation of $\overbrace{RR\cdots R}^{n+1}\overbrace{LL\cdots L}^{n-1}$.
- As the correspondence is one-to-one between permutations of $\overbrace{RR\cdots R}^{n+1}\overbrace{LL\cdots L}^{n-1}$ and illegal permutations (why?), there are $\binom{2n}{n-1}$ illegal walks.