



Grade 6 Math Circles

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Number Systems and Bases



Numbers are very important. Numbers are a part of our daily lives. We use numbers to keep track of the time, count the money in our piggy banks, call and text our friends and family, the list goes on and on and on.

Number System: The mathematical notation for representing a set of numbers using digits or symbols

Base: The number of unique digits, including zero, used to represent numbers

In today's society, our knowledge of numbers has grown so much that we have multiple number systems. But where did they all originate from? There are many ancient number systems but today we will learn about the ancient Egyptian number system.

Ancient Egyptian Number System

Egyptian hieroglyphs was the formal writing system used by the ancient Egyptians. Not only did they have hieroglyphs for words, they had hieroglyphs for numbers as well. Similar to our decimal system, the ancient Egyptians also used a base 10 number system to represent numbers as shown in the figure below.

Line	Heel bone	Coil of rope	Water lily	Finger	Tadpole	Man
	∩	⌚	☐		🐸	👤
1	10	100	1 000	10 000	100 000	1 000 000

Contrary to how we write left to right, the ancient Egyptians wrote their numbers from right to left! To write a number using the Egyptian hieroglyphs, we write the largest value first followed by smaller values in descending order from right to left.

For example, we would write 4622 as follows:



Example 1 Write 53 441 using the Egyptian hieroglyphs:

Example 2 Solve the following equation using Egyptian hieroglyphs:

$$\begin{array}{c}
 ||| \cap \cap \cap \cap \cap \cap \cap \cap \\
 ||| \cap \cap \cap \cap \cap \cap \cap \cap
 \end{array}
 +
 \begin{array}{c}
 || \cap \cap \cap \cap \cap \cap \\
 || \cap \cap \cap \cap \cap \cap
 \end{array}
 =$$

Decimal Numbers

As mentioned earlier, the **decimal number system** is the number we use to represent numbers in our lives. Similar to the ancient Egyptians, our decimal number system is a base 10 number system because we have 10 unique digits (0, 1, 2, 3, 4, 5, 6, 7, 8, 9) to represent numbers.

We can write numbers using these 10 digits or in words as shown in the example below.

$$4892 \implies \text{four thousand eight hundred and ninety two}$$

We can also write numbers using powers of 10! Here are a few powers of 10...

10^0	10^1	10^2	10^3	10^4	...
1	10	100	1 000	10 000	...

Below is an example of how we can write numbers in expanded form using powers of 10...

$$4892 = 4000 + 800 + 90 + 2$$

=

=

Example 1 Write 539 in expanded form:

Example 2 Write the following in standard form (our usual digit representation):

$$(3)(10^3) + (4)(10^2) + (7)(10^1) + (5)(10^0)$$

Binary Numbers

A _____ is a number that is represented by 0s and 1s. For example, 11, 101, 1011 and 101101 are all binary numbers. The **binary number system** is another commonly used number system. In fact, all of our electronic devices use binary numbers and we probably do not even know it! This is a base 2 number system because we represent our numbers using only two unique digits, 0 and 1.

Binary numbers are read a little differently.

$$101101 \implies \text{one-zero-one-one-zero-one}$$

Converting from Binary to Decimal

Earlier, we learned how to write decimal numbers in expanded form using powers of 10. Similar to that, we can write binary numbers using powers of 2!

2^0	2^1	2^2	2^3	2^4	2^5	2^6	2^7	...
1	2	4	8	16	32	64	128	...

Let's try an example and rewrite 101101...

$$101101 \longrightarrow$$

$$=$$

$$=$$

So, the binary number 101101 is equal to the decimal number _____.

Example Convert 111011 into a decimal number:

Converting from Decimal to Binary

So we have learned how to convert binary numbers into decimal numbers, but what about the other way around? We can convert decimal numbers into binary numbers using division!

Recall:

$$\begin{array}{r}
 \text{quotient} \rightarrow 57 \text{ dividend} \\
 \text{divisor} \rightarrow 3 \overline{) 172} \\
 \underline{150} \\
 22 \\
 \underline{21} \\
 \text{remainder} \rightarrow 1
 \end{array}$$

Suppose we want to convert 62 into a binary number. Since we are working in a base 2 number system, we will divide by 2.

Division	Remainder
$62 \div 2 = 31$	0

If we read the remainder column from the bottom to the top, the decimal number 62 is converted to the binary number _____. Don't believe it? We can check by converting our binary number back into a decimal number. Try it for yourself as an exercise!

Example Convert 37 from decimal to binary:

Hexadecimal Numbers

Here is another commonly used number system. The _____ is used in accessing computer memory, in defining colours and more! It is widely used by computer system designers and programmers. This is a base 16 number system and uses the following numbers and letters to represent numbers:

(0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F)

A-F represents the numbers 10-15 respectively. For example, 2A1C is a hexadecimal number.

Converting from Hexadecimal to Decimal

We can convert hexadecimal numbers into decimal numbers in the same way we convert binary numbers into decimal numbers except this time, we are working in base 16.

16^0	16^1	16^2	16^3	16^4	...
1	16	256	4096	65536	...

Rewrite A1C into a decimal number...

A1C \rightarrow

=

=

The hexadecimal number A1C is equal to the decimal number _____.

Example Convert 2C5 from hexadecimal to decimal:

Converting from Decimal to Hexadecimal

To convert a decimal number into a hexadecimal number, we can use the same method we used to convert decimal numbers into binary!

For an example, let's convert 11823 into a hexadecimal number. Since hexadecimal numbers are in a base 16 number system, we will be dividing by 16.

Division	Remainder
$11823 \div 16 = 738$	15 (F)

Reading the remainder column from the bottom to the top, the decimal number 11823 is converted to the hexadecimal number _____. Also like before, you check to see if this is correct by converting the hexadecimal number back into a decimal number.

Example Convert 2548 from decimal into hexadecimal:

Notation for Number Systems

There are so many different number systems with different bases, it can get a little confusing. How can we tell the difference between numbers in different bases? We include the base as a subscript when writing numbers.

Examples

- * The decimal number 435 is written as 435_{10}
- * The binary number 110110 is written as 110110_2
- * $101101_2 = 45_{10}$
- * $2A1C_{16} = 10780_{10}$

Other Base Number Systems

Decimal, binary, and hexadecimal number systems are the more commonly used systems that we use. However, there are many different base number systems! For example, you can have numbers in base 46. The ancient Babylonians used a base 60 number system!

So how do we work with these other base number systems? We can pretty much deal with any base number system the way we have been dealing with the binary, decimal, and hexadecimal number systems.

Converting from One Base Number to Another

Suppose we want to convert 1420_5 into a base 7 number. There is no immediate direct way of converting from base 5 to base 7. To do this, we do the following steps:

We can convert 1420_5 into a decimal number as seen below...

$$\begin{aligned} 1420_5 &= \\ &= \\ &= 235_{10} \end{aligned}$$

Next, convert 235_{10} into a base 7 number as we did for the previous few base number systems. Since we want our number to be in base 7, we will divide by 7!

Division	Remainder
$235 \div 7 = 33$	4

Reading the remainder column from the bottom to the top, the correct answer is _____.

Example Convert the 1010_3 into a base 6 number.

Problem Set

1. Write the following numbers using Egyptian hieroglyphs:

(a) 82 073

(b) 123 456

(c) 1 240 518

2. Solve the following using Egyptian hieroglyphs:

(a) =

(b) =

(c) =

3. What integers are allowed to be digits in the base 7 number system?

4. Write the following numbers in expanded form:

(a) 6221_{10}

(b) 101101_2

(c) $F4A_{16}$

(d) 24607_8

5. Write the following in standard form:

(a) $(2)(10^3) + (1)(10^2) + (0)(10^1) + (6)(10^0)$

(b) $(8)(16^2) + (11)(16^1) + (3)(16^0)$

(c) $(1)(2^4) + (1)(2^3) + (0)(2^2) + (1)(2^1) + (0)(2^0)$

(d) $(10)(18^3) + (6)(18^2) + (17)(18^1) + (2)(18^0)$

6. Why do you think we use a base 10 number system?

7. Convert the following numbers into a decimal number:

(a) 1011_2

(b) $1EA7_{16}$

(c) 3321_4

8. Convert the following numbers from decimal to binary:

(a) 88_{10}

(b) 524_{10}

(c) 1015_{10}

9. Convert the following numbers from decimal to hexadecimal:

(a) 488_{10}

(b) 10035_{10}

(c) 12451_{10}

10. Convert the following numbers into the base number given in parentheses:

(a) 122_3 (base 7)

(b) 842_9 (base 5)

(c) $EG42_{17}$ (base 15)

*11. We can use number systems to encrypt secret messages as well!

(a) Convert the alphabet into binary numbers. (ex. $A \rightarrow 1$, $B \rightarrow 10$, $C \rightarrow 11$, ...)

(b) Decrypt the message below:

```
"1-1100-1100 1111-10101-10010 100-10010-101-1-1101-10011 11-1-1110
  11-1111-1101-101 10100-10010-10101-101, 1001-110 10111-101
  1000-1-10110-101 10100-1000-101 11-1111-10101-10010-1-111-101
101000-1111 10000-10101-10010-10011-10101-101 10100-1000-101-1101."
```

- Walt Disney

12. Do the following binary calculations: (Hint:** It might help to convert the numbers into decimal numbers. But remember your answer must be a binary number!)

$$(a) \begin{array}{r} 0 \\ + 1 \\ \hline \end{array}$$

$$(b) \begin{array}{r} 10 \\ + 1 \\ \hline \end{array}$$

$$(c) \begin{array}{r} 11 \\ + 10 \\ \hline \end{array}$$

$$(d) \begin{array}{r} 111 \\ + 101 \\ \hline \end{array}$$

**13. (a) How many possible binary numbers with 6 digits are there?

(b) How many possible 6 digit binary numbers are there where the last digit is 1?

***14. For any given length $n \geq 2$

(a) How many possible binary numbers with n digits are there?

(b) How many possible numbers are there where the last digit is 1?